Detailed formulation of SCALE-RM

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September 10, 2018

Contents

Chapter 1

Introduction

1.1 What is SCALE

SCALE (Scalable Computing for Advanced Library and Environment)is a basic library of weather and climate models of the earth and planets intended for widespread use. The SCALE library was co-designed by computational science and computer science researchers.

Chapter 2

Governing equations

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2.1 Continuity equations

The continuity equations for each material can be described as the flux form:

$$\frac{\partial \rho q_d}{\partial t} + \nabla \cdot (\rho q_d \mathbf{u}) = \text{DIFF}[q_d] \tag{2.1}$$

$$\frac{\partial \rho q_v}{\partial t} + \nabla \cdot (\rho q_v \mathbf{u}) = S_v + \text{DIFF}[q_v] \tag{2.2}$$

$$\frac{\partial \rho q_l}{\partial t} + \nabla \cdot (\rho q_l \mathbf{u}) + \frac{\partial \rho q_l w_l}{\partial z} = S_l + \text{DIFF}[q_l] \tag{2.3}$$

$$\frac{\partial \rho q_s}{\partial t} + \nabla \cdot (\rho q_s \mathbf{u}) + \frac{\partial \rho q_s w_s}{\partial z} = S_s + \text{DIFF}[q_s] \tag{2.4}$$

The summation of the mass concentrations should be unit:

$$q_d + q_v + q_l + q_s = 1. \tag{2.5}$$

The source terms of water substances should satisfy the following relation:

$$S_v + S_l + S_s = 0. \tag{2.6}$$

The summation of Eqs.(2.1)-(2.4) gives the continuity equation of total density:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) + \frac{\partial \rho q_l w_l}{\partial z} + \frac{\partial \rho q_s w_s}{\partial z} = 0, \tag{2.7}$$

For this derivation, we assume that the operator DIFF [] is distributive. Using Eq.(2.5),

$$\begin{aligned} & \text{DIFF}[q_d] + \text{DIFF}[q_v] + \text{DIFF}[q_l] + \text{DIFF}[q_s] \\ = \; & \text{DIFF}[q_d + q_v + q_l + q_s] = \text{DIFF}[1] = 0 \end{aligned} \tag{2.8}$$

2.2 Momentum equations

The momentum equations for the gas, liquid, and solid material are described as

$$\frac{\partial \rho \left(q_d + q_v\right) \mathbf{u}}{\partial t} + \nabla \cdot \left[\rho \left(q_d + q_v\right) \mathbf{u} \otimes \mathbf{u}\right] \tag{2.9}$$

$$= -\nabla p - \left[\rho \left(q_d + q_v\right) g + \left(f_l + f_s\right)\right] \mathbf{e_z} + \mathbf{u} S_v + \mathrm{DIFF}\left[\left(q_d + q_v\right) \mathbf{u}\right] \tag{2.10}$$

$$\frac{\partial \rho q_l \mathbf{u}}{\partial t} + \nabla \cdot \left(\rho q_l \mathbf{u} \otimes \mathbf{u}\right) + \frac{\partial \rho q_l \mathbf{u} w_l}{\partial z} = -\left(\rho q_l g - f_l\right) \mathbf{e_z} + \mathbf{u} S_l + \mathrm{DIFF}\left[q_l \mathbf{u}\right] \tag{2.11}$$

$$\frac{\partial \rho q_s \mathbf{u}}{\partial t} + \nabla \cdot \left(\rho q_s \mathbf{u} \otimes \mathbf{u}\right) + \frac{\partial \rho q_s \mathbf{u} w_s}{\partial z} = -\left(\rho q_s g - f_s\right) \mathbf{e_z} + \mathbf{u} S_s + \mathrm{DIFF}\left[q_s \mathbf{u}\right] \tag{2.12}$$

The pressure is derived from the equation of state as

$$p = \rho \left(q_d R_d + q_v R_v\right) T. \tag{2.13}$$

The summation of Eqs.(2.10)-(2.12) gives the total momentum equation as

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \left(\rho \mathbf{u} \otimes \mathbf{u}\right) + \left(\frac{\partial \rho q_l w_l}{\partial z} + \frac{\partial \rho q_s w_s}{\partial z}\right) \mathbf{e_z} = -\nabla p - \rho g \mathbf{e_z} + \mathrm{DIFF}\left[\mathbf{u}\right] \tag{2.14}$$

Note that the drag forces by water loading does not appear in Eq.(2.14), because those term are cancelled out through the summation.

2.3 Thermodynamics equations

The equations of the internal energies are described as

$$\frac{\partial \rho (q_d e_d + q_v e_v)}{\partial t} + \nabla \cdot \left[\rho (q_d e_d + q_v e_v) \mathbf{u}\right] = -p \nabla \cdot \mathbf{u} + Q_d + Q_v + \mathrm{DIFF}\left[(q_d + q_v) T^*\right] \tag{2.15}$$

$$\frac{\partial \rho q_l e_l}{\partial t} + \nabla \cdot \left(\rho q_l e_l \mathbf{u}\right) + \frac{\partial \rho q_l e_l w_l}{\partial z} = Q_l + \mathrm{DIFF}\left[q_l T^*\right] \tag{2.16}$$

$$\frac{\partial \rho q_l e_s}{\partial t} + \nabla \cdot \left(\rho q_s e_s \mathbf{u}\right) + \frac{\partial \rho q_s e_s w_s}{\partial z} = Q_s + \mathrm{DIFF}\left[q_s T^*\right] \tag{2.17}$$

where T^* is some kind of potential temperature, discussed later. The internal energies are defined as

$$e_d = c_{vd} T \tag{2.18}$$

$$e_v = c_{vv} T \tag{2.19}$$

$$e_l = c_l T \tag{2.20}$$

$$e_s = c_s T, \tag{2.21}$$

The summation of Eqs.(2.15)-(2.17) gives the following internal energy equations:

$$\begin{aligned} & \frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \mathbf{u}) + \frac{\partial \rho q_l e_l w_l}{\partial z} + \frac{\partial \rho q_s e_s w_s}{\partial z} + p \nabla \cdot \mathbf{u} \\ = \ & Q + \mathrm{DIFF}\left[T^*\right] \end{aligned} \tag{2.22}$$

where

$$e = q_d e_d + q_v e_v + q_l e_l + q_s e_s, \tag{2.23}$$

and the total diabatic heating is described as

$$Q = Q_d + Q_v + Q_l + Q_s. \tag{2.24}$$

2.4 Conseptual seperation for solving the set of equations

Eqs.(2.2)-(2.4),(2.7),(2.14), and (2.13) with Eq.(2.22) are the complete set of equations. For solving them easily, we seperate the set of equations conceptually as

$$\frac{\partial \phi}{\partial t} = \left(\frac{\partial \phi}{\partial t}\right)_{dynamics} + \left(\frac{\partial \phi}{\partial t}\right)_{physics} \tag{2.25}$$

The falling proccess of liquid and solid waters, the source and sink process of water vapor, and the diabatic heating process for energy equations are treated as physical process, the others are treated as dynamical proccess.

According to this scheme, the dynamical process can be written as

$$\frac{\partial \rho q_v}{\partial t} + \nabla \cdot (\rho q_v \mathbf{u}) = 0 \tag{2.26}$$

$$\frac{\partial \rho q_l}{\partial t} + \nabla \cdot (\rho q_l \mathbf{u}) = 0 \tag{2.27}$$

$$\frac{\partial \rho q_s}{\partial t} + \nabla \cdot (\rho q_s \mathbf{u}) = 0 \tag{2.28}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2.29}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p - \rho g \mathbf{e_z} \tag{2.30}$$

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \mathbf{u}) + p \nabla \cdot \mathbf{u} = 0 \tag{2.31}$$

On the other hand, the physical processes are as follows:

$$\frac{\partial \rho q_v}{\partial t} = S_v + \mathrm{DIFF}\left[q_v\right] \tag{2.32}$$

$$\frac{\partial \rho q_l}{\partial t} + \frac{\partial \rho q_l w_l}{\partial z} = S_l + \mathrm{DIFF}\left[q_l\right] \tag{2.33}$$

$$\frac{\partial \rho q_s}{\partial t} + \frac{\partial \rho q_s w_s}{\partial z} = S_s + \mathrm{DIFF}\left[q_s\right] \tag{2.34}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho q_l w_l}{\partial z} + \frac{\partial \rho q_s w_s}{\partial z} = 0 \tag{2.35}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \frac{\partial \rho q_l \mathbf{u} w_l}{\partial z} + \frac{\partial \rho q_s \mathbf{u} w_s}{\partial z} = \mathrm{DIFF}\left[\mathbf{u}\right] \tag{2.36}$$

$$\frac{\partial \rho e}{\partial t} + \frac{\partial \rho q_l e_l w_l}{\partial z} + \frac{\partial \rho q_s e_s w_s}{\partial z} = Q + \mathrm{DIFF}\left[T^*\right] \tag{2.37}$$

2.5 Conservation of thermodynamics in the dynamical process

Equation (2.31) is not a complete flux form, because the internal energy itself is not conserved both in the Euler sense and in the Lagrangian sense. In this section, we consider the conservative quantity for thermodynamics equation.

In the dry atmosphere, the potential temperature for dry air, which is defined as

$$\theta_d = T\left(\frac{p_{00}}{p}\right)^{R_d/c_{pd}}, \tag{2.38}$$

is used as a conserved quantity it is conserved along the Lagrange trajectory c_{pd} R_d are the specific heats at constant pressure and However, it is no longer satisfied when the water substances are included.

Sinece Eq.(2.29) is equivallent to

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{u} = 0, \tag{2.39}$$

Equation (2.31) is

$$\rho\frac{de}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} = 0. \tag{2.40}$$

Dividing by ρ, this equation can be written as

$$\frac{de}{dt} + p\frac{d}{dt}\left(\frac{1}{\rho}\right) = 0. \tag{2.41}$$

Substiting Eq.(2.13) into Eq.(2.41),

$$\frac{dq_d c_{vd} T}{dt} + p\frac{d}{dt}\left[\frac{q_d R_d T}{p}\right] + \frac{dq_v c_{vv} T}{dt} + p\frac{d}{dt}\left[\frac{q_v R_v T}{p}\right] + \frac{dq_l c_l T}{dt} + \frac{dq_s c_s T}{dt} = 0 \tag{2.42}$$

Since Eqs.(2.26)-(2.29) give

$$\frac{dq_d}{dt} = \frac{dq_v}{dt} = \frac{dq_l}{dt} = \frac{dq_s}{dt} = 0, \tag{2.43}$$

Equation (2.42) gives the following form:

$$q_d \left[\frac{dc_{vd}T}{dt} + p\frac{d}{dt}\left[\frac{R_d T}{p}\right]\right] + q_v \left[\frac{dc_{vv}T}{dt} + p\frac{d}{dt}\left[\frac{R_v T}{p}\right]\right]$$
$$+q_l\frac{dc_l T}{dt} + q_s\frac{dc_s T}{dt} = 0 \tag{2.44}$$

Dividing this equation by T,

$$q_d\left[c_{pd}\frac{1}{T}\frac{dT}{dt} + R_d p\frac{d}{dt}\left(\frac{1}{p}\right)\right] + q_v\left[c_{pv}\frac{1}{T}\frac{dT}{dt} + R_v p\frac{d}{dt}\left(\frac{1}{p}\right)\right]$$
$$+q_l c_l\frac{1}{T}\frac{dT}{dt} + q_s c_s\frac{1}{T}\frac{dT}{dt} = 0 \tag{2.45}$$
$$q_d c_{pd}\left[\frac{d\ln T}{dt} + \frac{R_d}{c_{pd}}\frac{d}{dt}\left[\ln\left(\frac{1}{p}\right)\right]\right] + q_v c_{pv}\left[\frac{d\ln T}{dt} + \frac{R_v}{c_{pv}}\frac{d}{dt}\left[\ln\left(\frac{1}{p}\right)\right]\right]$$
$$+q_l c_l\frac{d\ln T}{dt} + q_s c_s\frac{d\ln T}{dt} = 0 \tag{2.46}$$
$$q_d c_{pd}\frac{d\ln\theta_d}{dt} + q_v c_{pv}\frac{d\ln\theta_v}{dt} + q_l c_l\frac{d\ln T}{dt} + q_s c_s\frac{d\ln T}{dt} = 0 \tag{2.47}$$
$$\frac{d}{dt}\left[\ln\left(\theta_d^{q_d c_{pd}}\theta_v^{q_v c_{pv}}T^{q_l c_l}T^{q_s c_s}\right)\right] = 0 \tag{2.48}$$

Thus,

$$\frac{d}{dt}\left[\theta_d^{q_d c_{pd}}\theta_v^{q_v c_{pv}}T^{q_l c_l}T^{q_s c_s}\right] = 0 \tag{2.49}$$

Thus, the following quantity is conserved along the flow trajectory;

$$\Theta = \theta_d^{q_d c_{pd}}\theta_v^{q_v c_{pv}}T^{q_l c_l}T^{q_s c_s} \tag{2.50}$$

where θ_v is the potential temperature for water vapor, defined as

$$\theta_v = T\left(\frac{p_{00}}{p}\right)^{R_v/c_{pv}} \tag{2.51}$$

The equation of state has the following expression using Θ.

$$\Theta = T^{q_d c_{pd}}\left(\frac{p_{00}}{p}\right)^{q_d R_d} T^{q_v c_{pv}}\left(\frac{p_{00}}{p}\right)^{q_v R_v} T^{q_l c_l} + T^{q_s c_s} \tag{2.52}$$
$$= T^{q_d c_{pd}+q_v c_{pv}+q_l c_l+q_s c_s}\left(\frac{p_{00}}{p}\right)^{q_d R_d + q_v R_v} \tag{2.53}$$
$$= T^{c_p^*}\left(\frac{p_{00}}{p}\right)^{R^*}, \tag{2.54}$$

where

$$c_p^* \equiv q_d c_{pd} + q_v c_{pv} + q_l c_l + q_s c_s \tag{2.55}$$
$$R^* \equiv q_d R_d + q_v R_v \tag{2.56}$$

We define a new potential temperature

$$\theta \equiv \Theta^{1/c_p^*} = T\left(\frac{p_{00}}{p}\right)^{R^*/c_p^*} \quad (2.57)$$

The pressure expression is derived diagnostically as follows:

$$p = \rho(q_d R_d + q_v R_v)\theta\left(\frac{p}{p_{00}}\right)^{\frac{R^*}{c_p^*}} \quad (2.58)$$

$$p^{1-\frac{R^*}{c_p^*}} = \rho R^* \theta\left(\frac{1}{p_{00}}\right)^{\frac{R^*}{c_p^*}} \quad (2.59)$$

$$p = p_{00}\left(\frac{\rho\theta R^*}{p_{00}}\right)^{\frac{c_p^*}{c_p^*-R^*}} \quad (2.60)$$

Note that

$$\frac{d\theta}{dt} = \frac{1}{a}\Theta^{1/a-1}\frac{d\Theta}{dt} = 0 \quad (2.61)$$

Therefore, $\rho\theta$ can be employed for the prognostic varaiable!

Figure 2.1(a) gives the vertical profile of the temperature in the U.S.standard atmosphere and Fig.2.1(b) shows the vetical profiles of θ/θ_d under this temperature condition when we assume that q_v is mass concentration of water vapor at the saturation, $q_l + q_s$ gives 0.0, 0.01, 0.02, and 0.04. The differnce between θ and θ_d becomes larger with the height and it may not be negligible.

2.6 Diabatic heating in the physical process

If the prognostic variable for thermodynamics is changed from the internal energy to the newly defined potential temperature θ, the diabatic heating in Eq.(2.37) should be modified. Through the manupulation from Eq.(2.40) to Eq.(2.48), Eq.(2.37) without turbulence term can be written as

$$\frac{d\ln\Theta}{dt} = \frac{Q}{\rho T} \quad (2.62)$$

On the other hand, Eq.(2.61) gives

$$\frac{d\theta}{dt} = \frac{1}{c_p^*}\Theta^{1/a}\frac{d\ln\Theta}{dt} \quad (2.63)$$

Substituting Eq.(2.62) into Eq.(2.63),

$$\frac{d\theta}{dt} = \frac{1}{c_p^*}\left(\frac{p}{p_{00}}\right)^{\frac{R^*}{c_p^*}}\frac{Q}{\rho} \quad (2.64)$$

2.7 Summary of equations in the dynamical process and physical process

2.7.1 The dynamical process

$$\frac{\partial \rho q_v}{\partial t} + \nabla \cdot (\rho q_v \mathbf{u}) = \left(\frac{\partial \rho q_v}{\partial t}\right)_{physics} \tag{2.65}$$

$$\frac{\partial \rho q_l}{\partial t} + \nabla \cdot (\rho q_l \mathbf{u}) = \left(\frac{\partial \rho q_l}{\partial t}\right)_{physics} \tag{2.66}$$

$$\frac{\partial \rho q_s}{\partial t} + \nabla \cdot (\rho q_s \mathbf{u}) = \left(\frac{\partial \rho q_s}{\partial t}\right)_{physics} \tag{2.67}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \left(\frac{\partial \rho}{\partial t}\right)_{physics} \tag{2.68}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p - \rho g \mathbf{e_z} + \left(\frac{\partial \rho \mathbf{u}}{\partial t}\right)_{physics} \tag{2.69}$$

$$\frac{\partial \rho \theta}{\partial t} + \nabla \cdot (\rho \theta \mathbf{u}) = \left(\frac{\partial \rho \theta}{\partial t}\right)_{physics} \tag{2.70}$$

$$p = p_{00} \left(\frac{\rho \theta R^*}{p_{00}}\right)^{\frac{c_p^*}{c_p^* - R^*}} \tag{2.71}$$

where

$$c_p^* \equiv q_d c_{pd} + q_v c_{pv} + q_l c_l + q_s c_s \tag{2.72}$$

$$R^* \equiv q_d R_d + q_v R_v \tag{2.73}$$

2.7.2 The physical process

$$\left(\frac{\partial \rho q_v}{\partial t}\right)_{physics} = S_v + \mathrm{DIFF}\left[q_v\right] \tag{2.74}$$

$$\left(\frac{\partial \rho q_l}{\partial t}\right)_{physics} = -\frac{\partial \rho q_l w_l}{\partial z} + S_l + \mathrm{DIFF}\left[q_l\right] \tag{2.75}$$

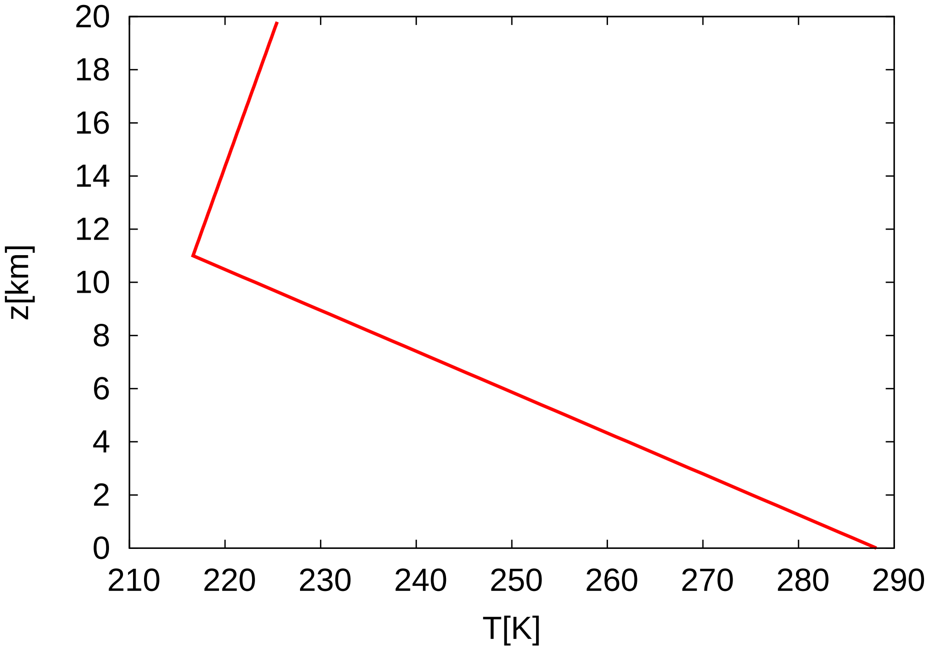
$$\left(\frac{\partial \rho q_s}{\partial t}\right)_{physics} = -\frac{\partial \rho q_s w_s}{\partial z} + S_s + \mathrm{DIFF}\left[q_s\right] \tag{2.76}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{physics} = -\frac{\partial \rho q_l w_l}{\partial z} - \frac{\partial \rho q_s w_s}{\partial z} \tag{2.77}$$

$$\left(\frac{\partial \rho \mathbf{u}}{\partial t}\right)_{physics} = -\frac{\partial \rho q_l \mathbf{u} w_l}{\partial z} - \frac{\partial \rho q_s \mathbf{u} w_s}{\partial z} + \mathrm{DIFF}\left[\mathbf{u}\right] \tag{2.78}$$

$$\left(\frac{\partial \rho \theta}{\partial t}\right)_{physics} = \frac{1}{c_p^*}\left(\frac{p}{p_{00}}\right)^{\frac{R^*}{c_p^*}} \left[Q - \frac{\partial \rho q_l e_l w_l}{\partial z} - \frac{\partial \rho q_s e_s w_s}{\partial z}\right] + \mathrm{DIFF}\left[\theta\right] \tag{2.79}$$

(a)



(b)

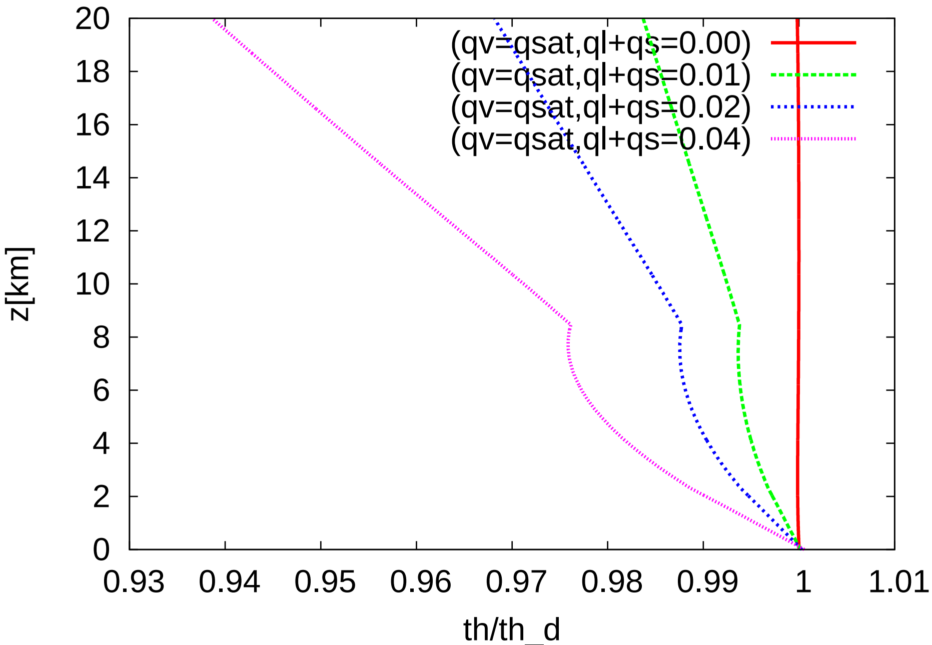


Figure 2.1: Thee vertical profile of (a) U.S. standard atmosphere, (b) Several profiles of θ/θ_d.

Chapter 3

Discretization of dynamics

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3.1 Temporal integration scheme

3.1.1 Runge-Kutta schemes

For the time integration of Eqs.(2.68)-(2.40), we adopt the full explicit scheme with the p step Runge-Kutta scheme.

$$\phi_0^* = \phi^t \tag{3.1}$$
$$k_1 = f(\phi^t) \tag{3.2}$$
$$k_2 = f(\phi^t + k_1 \Delta t \alpha_1) \tag{3.3}$$
$$\dots$$
$$k_p = f(\phi^t + k_{p-1} \Delta t \alpha_{p-1}) \tag{3.4}$$
$$\phi^{t+\Delta t} = \phi^t + \Delta t \sum_p \beta_p k_p. \tag{3.5}$$

The 3 and 4 step Runge-Kutta scheme are implemented.

The Heun's three step scheme

$$k_1 = f(\phi^n), \tag{3.6}$$
$$k_2 = f\left(\phi^n + \frac{1}{3}\Delta t k_1\right), \tag{3.7}$$
$$k_3 = f\left(\phi^n + \frac{2}{3}\Delta t k_2\right), \tag{3.8}$$
$$\phi^{n+1} = \phi^n + \frac{1}{4}\Delta t(k_1 + 3k_3). \tag{3.9}$$

The Kutta's three step scheme

$$k_1 = f(\phi^n), \tag{3.10}$$
$$k_2 = f\left(\phi^n + \frac{1}{2}\Delta t k_1\right), \tag{3.11}$$
$$k_3 = f\left(\phi^n - \Delta t k_1 + 2\Delta t k_2\right), \tag{3.12}$$
$$\phi^{n+1} = \phi^n + \frac{1}{6}\Delta t(k_1 + 4k_2 + k_3). \tag{3.13}$$

The Wicker and Skamarock (2002)'s three step scheme

$$k_1 = f(\phi^n), \tag{3.14}$$
$$k_2 = f\left(\phi^n + \frac{1}{3}\Delta t k_1\right), \tag{3.15}$$
$$k_3 = f\left(\phi^n + \frac{1}{2}\Delta t k_2\right), \tag{3.16}$$
$$\phi^{n+1} = \phi^n + \Delta t k_3. \tag{3.17}$$

The four step scheme

$$k_1 = f(\phi^n), \tag{3.18}$$
$$k_2 = f\left(\phi^n + \frac{1}{2}\Delta t k_1\right), \tag{3.19}$$
$$k_3 = f\left(\phi^n + \frac{1}{2}\Delta t k_2\right), \tag{3.20}$$
$$k_4 = f\left(\phi^n + \Delta t k_3\right), \tag{3.21}$$
$$\phi^{n+1} = \phi^n + \frac{1}{6}\Delta t(k_1 + 2k_2 + 2k_3 + k_4). \tag{3.22}$$

The forward-backward scheme

In the short time step, the momentums are updated first and then density is updated with the updated momentums.

$$\rho u_i^{n+1} = \rho u_i^n + \Delta t f_{\rho u_i}(\rho^n), \tag{3.23}$$
$$\rho^{n+1} = \rho^n + \Delta t f_\rho(\rho u_i^{n+1}). \tag{3.24}$$

3.1.2 Numerical stability

A fully compressive equations of a acoustic mode is considered. The continuous and momentum equations is the followings:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u_i}{\partial x_i} \tag{3.25}$$

$$\frac{\partial \rho u_i}{\partial t} = -\frac{\partial p}{\partial x_i} \tag{3.26}$$

$$p = p_0 \left(\frac{R\rho\theta}{p_0} \right)^{c_p/c_v}, \tag{3.27}$$

here the potential temperature θ is assumed to be constant.

In order to analize the numerical stability of equation, the equation of the state is linearized.

$$p \approx \bar{p} + c^2 \rho', \tag{3.28}$$

where c is the sound speed: $c^2 = \frac{c_p \bar{p}}{c_v \bar{\rho}}$.

We descritize the governing equation with the 2th order central difference.

$$\left.\frac{\partial \rho}{\partial t}\right|_{i,j,k} = -\frac{U_{i+1/2} - U_{i-1/2}}{\Delta x} - \frac{V_{j+1/2} - V_{j-1/2}}{\Delta y} - \frac{W_{k+1/2} - W_{k-1/2}}{\Delta z} \tag{3.29}$$

$$\left.\frac{\partial U}{\partial t}\right|_{i+1/2} = -c^2 \frac{\rho_{i+1} - \rho_i}{\Delta x} \tag{3.30}$$

$$\left.\frac{\partial V}{\partial t}\right|_{j+1/2} = -c^2 \frac{\rho_{j+1} - \rho_j}{\Delta y} \tag{3.31}$$

$$\left.\frac{\partial W}{\partial t}\right|_{i+1/2} = -c^2 \frac{\rho_{k+1} - \rho_k}{\Delta z}, \tag{3.32}$$

where U, V, and W is the momentum at the stagared grid point in x, y, and z direction, respectively.

The error of the spatial differenece of a wavenumber k component $\hat{\phi}_k$ is $\{\exp(ik\Delta x) - 1\}\,\hat{\phi}$, and the error of 2-grid mode is the largest: $\exp(i\pi) - 1 = -2$.

The temporal differential of the 2-grid mode is

$$\frac{\partial \rho}{\partial t} = -\frac{1 - \exp(-i\pi)}{\Delta x} U - \frac{1 - \exp(-i\pi)}{\Delta y} V - \frac{1 - \exp(-i\pi)}{\Delta z} W \tag{3.33}$$

$$\frac{\partial U}{\partial t} = -c^2 \frac{\exp(i\pi) - 1}{\Delta x} \rho \tag{3.34}$$

$$\frac{\partial V}{\partial t} = -c^2 \frac{\exp(i\pi) - 1}{\Delta y} \rho \tag{3.35}$$

$$\frac{\partial W}{\partial t} = -c^2 \frac{\exp(i\pi) - 1}{\Delta z} \rho. \tag{3.36}$$

The mode of which the U, V and W has the same phase is the most unstable:

$$\frac{\partial \rho}{\partial t} = -3 \frac{1 - \exp(-i\pi)}{\Delta x} U \tag{3.37}$$

$$\frac{\partial U}{\partial t} = -c^2 \frac{\exp(i\pi) - 1}{\Delta x} \rho \tag{3.38}$$

Writing matrix form,

$$\begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial U}{\partial t} \end{pmatrix} = D \begin{pmatrix} \rho \\ U \end{pmatrix}, \tag{3.39}$$

where

$$D = \begin{pmatrix} 0 & -\frac{6}{\Delta x} \\ \frac{2c^2}{\Delta x} & 0 \end{pmatrix}. \tag{3.40}$$

The Euler scheme

With the Euler scheme,

$$\phi^{n+1} = \phi^n + \Delta t f(\phi^n) \tag{3.41}$$

The A is the matrix representing the time step, then

$$A = I + dtD, \tag{3.42}$$

$$= \begin{pmatrix} 1 & -6\frac{\Delta t}{\Delta x} \\ \frac{2c^2 \Delta t}{\Delta x} & 1 \end{pmatrix}. \tag{3.43}$$

The eigen value of A is larger than 1, and the Euler scheme is instable for any Δt.

The second step Runge-Kutta scheme

The Heun's second step Runge-Kutta scheme is

$$k_1 = f(\phi^n), \tag{3.44}$$

$$k_2 = f(\phi^n + \Delta t k_1), \tag{3.45}$$

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{2}(k_1 + k_2). \tag{3.46}$$

$$A = I + \frac{\Delta t}{2}(K_1 + K_2), \tag{3.47}$$

$$K_1 = D, \tag{3.48}$$

$$K_2 = D(I + \Delta t K_1). \tag{3.49}$$

After all,

$$A = \begin{pmatrix} 1 - 6\nu^2 & -\frac{6\Delta t}{\Delta x} \\ \frac{2c^2 \Delta t}{\Delta x} & 1 - 6\nu^2 \end{pmatrix}, \tag{3.50}$$

where ν is the Courant number for the sound speed: $\frac{c\Delta t}{\Delta x}$. The eigen value of A is larger than 1, and the Euler scheme is instable for any Δt.

The third step Runge-Kutta scheme

With the Heun's third step Runge-Kutta scheme, the matrix A is written by

$$A = I + \frac{\Delta t}{4}(K_1 + 4K_3), \tag{3.51}$$

$$= \begin{pmatrix} 1 - 6\nu^2 & -\frac{6\Delta t}{\Delta x}(1 - 2\nu^2) \\ \frac{2c^2 \Delta t}{\Delta x}(1 - 2\nu^2) & 1 - 6\nu^2 \end{pmatrix}, \tag{3.52}$$

where

$$K_1 = D, \tag{3.53}$$

$$K_2 = D(I + \frac{\Delta t}{3} K_1), \tag{3.54}$$

$$K_3 = D(I + \frac{2\Delta t}{3} K_2). \tag{3.55}$$

The condition that all the eigen values are less than or equal to 1 is

$$\nu \leq \frac{1}{2}. \tag{3.56}$$

In the Kutta's three step Runge-Kutta scheme, the matrix A is

$$A = I + \frac{\Delta t}{6}(K_1 + 4K_2 + K_3), \tag{3.57}$$

where

$$K_1 = D, \tag{3.58}$$

$$K_2 = D\left(I + \frac{\Delta t}{2} K_1\right), \tag{3.59}$$

$$K_3 = D\left(I - \Delta t K_1 + 2\Delta t K_2\right). \tag{3.60}$$

It is the idential as that in the Heun's scheme (eq. 3.52). Thus, the stable condition is the same (eq. 3.56).

The Wicker and Skamarock (2002)'s Runge-Kutta scheme is described as

$$A = I + \Delta t K_3, \tag{3.61}$$

$$K_1 = D, \tag{3.62}$$

$$K_2 = D\left(I + \frac{\Delta t}{3} K_1\right), \tag{3.63}$$

$$K_3 = D\left(I + \frac{\Delta t}{2} K_2\right). \tag{3.64}$$

The A and the consequent stable condition are the identical as the above two schemes.

The four step Runge-Kutta scheme

The matrix A is

$$A = I + \frac{\Delta t}{6}(K_1 + 2K_2 + 2K_3 + K_4), \tag{3.65}$$

$$= \begin{pmatrix} 1 - 6\nu^2 + 6\nu^4 & -\frac{6\Delta t}{\Delta x}(1 - 2\nu^2) \\ \frac{2c^2\Delta t}{\Delta x}(1 - 2\nu^2) & 1 - 6\nu^2 + 6^4 \end{pmatrix}, \tag{3.66}$$

where

$$K_1 = D, \tag{3.67}$$

$$K_2 = D\left(I + \frac{\Delta t}{2}K_1\right), \tag{3.68}$$

$$K_3 = D\left(I + \frac{\Delta t}{2}K_2\right), \tag{3.69}$$

$$K_4 = D(I + \Delta t K_3). \tag{3.70}$$

The condition for stability is

$$\nu \leq \frac{\sqrt{6}}{3}. \tag{3.71}$$

The number of floating opint operations with the four step Runge-Kutta scheme is about $4/3$ times larger than that with the three step scheme. Howere, the time step can be $2\sqrt{6}/3$ larger than that in the three step scheme. Since $2\sqrt{6}/3 > 4/3$, the four step Runge-Kutta scheme is more cost effective than the three step scheme in terms of numerical stability.

The forward-backward scheme

The stabitlity condition is

$$\nu \leq \frac{1}{\sqrt{3}}. \tag{3.72}$$

The forward-backward scheme can be used in each step in the Runge-Kutta schemes. The stability conditions are the followings:

The second step RK scheme

$$\nu \leq \frac{1}{\sqrt{3}}. \tag{3.73}$$

The Heun's three step RK scheme

$$\nu \leq \frac{1}{2}. \tag{3.74}$$

The Kutta's three step RK scheme

$$\nu \leq \frac{1}{2}. \tag{3.75}$$

TheWicker and Skamarock (2002)'s three step RK scheme

$$\nu \leq \frac{\sqrt{6}}{4}. \tag{3.76}$$

The four step RK scheme

$$\nu \leq 0.66 \tag{3.77}$$

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3.2 Spatial descretization

We employ the Arakawa-C staggered grid with the 3-dimensional momentum $(\rho u, \rho v, \rho w)$, density (ρ) and mass-weighted potentail temperature$(\rho\theta)$ as the prognostic variables. Figure 3.1(a) shows the structure of the control volume for the mass, indicating the location of each of prognostic variables. Conceptually, we use the 4th order central difference scheme for the advection or convection terms and the 2nd order central difference scheme for the other terms. Before the descretization of differential equations, we should diagnose several quantities from the prognostic variables.

Full-level pressure and potential temperature

$$p_{i,j,k} = p_{00} \left[\frac{(\rho\theta)_{i,j,k} R^*}{p_{00}} \right]^{\frac{c_p^*}{c_p^* - R^*}} \tag{3.78}$$

$$\theta_{i,j,k} = \frac{(\rho\theta)_{i,j,k}}{\rho_{i,j,k}} \tag{3.79}$$

$$\tag{3.80}$$

Half-level density

$$\overline{\rho}_{i+\frac{1}{2},j,k} = \frac{\rho_{i+1,j,k} + \rho_{i,j,k}}{2} \tag{3.81}$$

$$\overline{\rho}_{i,j+\frac{1}{2},k} = \frac{\rho_{i,j+1,k} + \rho_{i,j,k}}{2} \tag{3.82}$$

$$\overline{\rho}_{i,j,k+\frac{1}{2}} = \frac{\Delta z_k \rho_{i,j,k+1} + \Delta z_{k+1} \rho_{i,j,k}}{\Delta z_k + \Delta z_{k+1}} \tag{3.83}$$

Half-level velocity

$$\overline{u}_{i+\frac{1}{2},j,k} = \frac{(\rho u)_{i+\frac{1}{2},j,k}}{\overline{\rho}_{i+\frac{1}{2},j,k}} \tag{3.84}$$

$$\overline{v}_{i,j+\frac{1}{2},k} = \frac{(\rho v)_{i,j+\frac{1}{2},k}}{\overline{\rho}_{i,j+\frac{1}{2},k}} \tag{3.85}$$

$$\overline{w}_{i,j,k+\frac{1}{2}} = \frac{(\rho w)_{i,j,k+\frac{1}{2}}}{\overline{\rho}_{i,j,k+\frac{1}{2}}} \tag{3.86}$$

Full-level velocity

$$\overline{u}_{i,j,k} = \frac{(\rho u)_{i+\frac{1}{2},j,k} + (\rho u)_{i-\frac{1}{2},j,k}}{2\rho_{i,j,k}} \tag{3.87}$$

$$\overline{v}_{i,j,k} = \frac{(\rho v)_{i,j+\frac{1}{2},k} + (\rho v)_{i,j-\frac{1}{2},k}}{2\rho_{i,j,k}} \tag{3.88}$$

$$\overline{w}_{i,j,k} = \frac{(\rho w)_{i,j,k+\frac{1}{2}} + (\rho w)_{i,j,k-\frac{1}{2}}}{2\rho_{i,j,k}} \tag{3.89}$$

3.2.1 Continuity equation

$$\left(\frac{\partial\rho}{\partial t}\right)_{i,j,k} = -\frac{(\rho u)_{i+\frac{1}{2},j,k} - (\rho u)_{i-\frac{1}{2},j,k}}{\Delta x} - \frac{(\rho v)_{i,j+\frac{1}{2},k} - (\rho v)_{i,j-\frac{1}{2},k}}{\Delta y} - \frac{(\rho w)_{i,j,k+\frac{1}{2}} - (\rho w)_{i,j,k-\frac{1}{2}}}{\Delta z} \quad (3.90)$$

3.2.2 Momentum equations

Figure 3.1(a) shows the structure of the control volume for the momentum in the x direction. The momentum equation is descretized as

$$\left(\frac{\partial\rho u}{\partial t}\right)_{i+\frac{1}{2},j,k} = -\frac{\overline{(\rho u)}_{i+1,j,k}\overline{u}_{i+1,j,k} - \overline{(\rho u)}_{i,j,k}\overline{u}_{i,j,k}}{\Delta x} - \frac{\overline{(\rho u)}_{i+\frac{1}{2},j+\frac{1}{2},k}\overline{v}_{i+\frac{1}{2},j+\frac{1}{2},k} - \overline{(\rho u)}_{i+\frac{1}{2},j-\frac{1}{2},k}\overline{v}_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta y} - \frac{\overline{(\rho u)}_{i+\frac{1}{2},j,k+\frac{1}{2}}\overline{w}_{i+\frac{1}{2},j,k+\frac{1}{2}} - \overline{(\rho u)}_{i+\frac{1}{2},j,k-\frac{1}{2}}\overline{w}_{i+\frac{1}{2},j,k-\frac{1}{2}}}{\Delta z} - \frac{p_{i+1,j,k} - p_{i,j,k}}{\Delta x}, \quad (3.91)$$

where

$$\overline{(\rho u)}_{i,j,k} = \frac{-(\rho u)_{i+\frac{3}{2},j,k} + 7(\rho u)_{i+\frac{1}{2},j,k} + 7(\rho u)_{i-\frac{1}{2},j,k} - (\rho u)_{i-\frac{3}{2},j,k}}{12} \quad (3.92)$$

$$\overline{(\rho u)}_{i+\frac{1}{2},j+\frac{1}{2},k} = \frac{-(\rho u)_{i+\frac{1}{2},j+2,k} + 7(\rho u)_{i+\frac{1}{2},j+1,k} + 7(\rho u)_{i+\frac{1}{2},j,k} - (\rho u)_{i+\frac{1}{2},j-1,k}}{12} \quad (3.93)$$

$$\overline{(\rho u)}_{i+\frac{1}{2},j,k+\frac{1}{2}} = \frac{-(\rho u)_{i+\frac{1}{2},j,k+2} + 7(\rho u)_{i+\frac{1}{2},j,k+1} + 7(\rho u)_{i+\frac{1}{2},j,k} - (\rho u)_{i+\frac{1}{2},j,k-1}}{12} \quad (3.94)$$

and the velocities at the cell wall for the staggered control volume to x direction are defined as

$$\overline{u}_{i,j,k} = \frac{\overline{u}_{i+\frac{1}{2},j,k} + \overline{u}_{i-\frac{1}{2},j,k}}{2} \quad (3.95)$$

$$\overline{v}_{i+\frac{1}{2},j+\frac{1}{2},k} = \frac{\overline{v}_{i,j+\frac{1}{2},k} + \overline{v}_{i+1,j+\frac{1}{2},k}}{2} \quad (3.96)$$

$$\overline{w}_{i+\frac{1}{2},j,k+\frac{1}{2}} = \frac{\overline{w}_{i,j,k+\frac{1}{2}} + \overline{w}_{i+1,j,k+\frac{1}{2}}}{2} \quad (3.97)$$

In this form, the 4th order accuracy is guaranteed on the condition of the constant velocity.

The momentum equations in the y and z directions are descretized in the same way:

$$\left(\frac{\partial \rho v}{\partial t}\right)_{i,j+\frac{1}{2},k} = -\frac{\overline{(\rho v)}_{i+\frac{1}{2},j+\frac{1}{2},k}\overline{u}_{i+\frac{1}{2},j+\frac{1}{2},k} - \overline{(\rho v)}_{i-\frac{1}{2},j+\frac{1}{2},k}\overline{u}_{i-\frac{1}{2},j+\frac{1}{2},k}}{\Delta x} - \frac{\overline{(\rho v)}_{i,j+1,k}\overline{v}_{i,j+1,k} - \overline{(\rho v)}_{i,j,k}\overline{v}_{i,j,k}}{\Delta y} - \frac{\overline{(\rho v)}_{i,j+\frac{1}{2},k+\frac{1}{2}}\overline{v}_{i,j+\frac{1}{2},k+\frac{1}{2}} - \overline{(\rho v)}_{i,j+\frac{1}{2},k-\frac{1}{2}}\overline{v}_{i,j+\frac{1}{2},k-\frac{1}{2}}}{\Delta z} - \frac{p_{i,j+1,k} - p_{i,j,k}}{\Delta y}, \tag{3.98}$$

$$\left(\frac{\partial \rho w}{\partial t}\right)_{i,j,k+\frac{1}{2}} = -\frac{\overline{(\rho w)}_{i+\frac{1}{2},j,k+\frac{1}{2}}\overline{u}_{i+\frac{1}{2},j,k+\frac{1}{2}} - \overline{(\rho w)}_{i-\frac{1}{2},j,k+\frac{1}{2}}\overline{u}_{i-\frac{1}{2},j,k+\frac{1}{2}}}{\Delta x} - \frac{\overline{(\rho w)}_{i,j+\frac{1}{2},k+\frac{1}{2}}\overline{w}_{i,j+\frac{1}{2},k+\frac{1}{2}} - \overline{(\rho w)}_{i,j-\frac{1}{2},k+\frac{1}{2}}\overline{w}_{i,j-\frac{1}{2},k+\frac{1}{2}}}{\Delta y} - \frac{\overline{(\rho w)}_{i,j,k+1}\overline{w}_{i,j,k+1} - \overline{(\rho w)}_{i,j,k}\overline{w}_{i,j,k}}{\Delta z} - \frac{p_{i,j,k+1} - p_{i,j,k}}{\Delta z} - \overline{\rho}_{i,j,k+\frac{1}{2}}g \tag{3.99}$$

Pressure

Since the pressure pertubation is much smaller than the absolute value of the pressure, truncation error of floating point value is relatively large and its precision could become smaller. Therefore, the pressure gradient terms are calculated from the deviation from reference pressure field satisfing the hydrostatic balance. Additionally, the calculation of the pressure (eq. 2.60) is linearlized avoiding a power calculation, which numerically costs expensive.

$$p \approx \bar{p} + \frac{c_p^*}{R^* C_v^*}\left(\frac{\rho\theta R^*}{p_{00}}\right)^{\frac{R^*}{c_v^*}}\{\rho\theta - \overline{\rho\theta}\} = \bar{p} + \frac{c_p^*}{c_v^*}\frac{\bar{p}}{\overline{\rho\theta}}(\rho\theta)' \tag{3.100}$$

$$p - p_{\mathrm{ref}} = \bar{p} - p_{\mathrm{ref}} + \frac{c_p^*}{c_v^*}\frac{\bar{p}}{\overline{\rho\theta}}(\rho\theta)' \tag{3.101}$$

3.2.3 Energy equation

$$\left(\frac{\partial \rho\theta}{\partial t}\right)_{i,j,k} = -\frac{(\rho u)_{i+\frac{1}{2},j,k}\overline{\theta}_{i+\frac{1}{2},j,k} - (\rho u)_{i-\frac{1}{2},j,k}\overline{\theta}_{i-\frac{1}{2},j,k}}{\Delta x} - \frac{(\rho v)_{i,j+\frac{1}{2},k}\overline{\theta}_{i,j+\frac{1}{2},k} - (\rho v)_{i,j-\frac{1}{2},k}\overline{\theta}_{i,j-\frac{1}{2},k}}{\Delta y} - \frac{(\rho w)_{i,j,k+\frac{1}{2}}\overline{\theta}_{i,j,k+\frac{1}{2}} - (\rho w)_{i,j,k-\frac{1}{2}}\overline{\theta}_{i,j,k-\frac{1}{2}}}{\Delta z} \tag{3.102}$$

where

$$\overline{\theta}_{i+\frac{1}{2},j,k} = \frac{-\theta_{i+2,j,k} + 7\theta_{i+1,j,k} + 7\theta_{i,j,k} - \theta_{i-1,j,k}}{12} \tag{3.103}$$

$$\overline{\theta}_{i,j+\frac{1}{2},k} = \frac{-\theta_{i,j+2,k} + 7\theta_{i,j+1,k} + 7\theta_{i,j,k} - \theta_{i,j-1,k}}{12} \tag{3.104}$$

$$\overline{\theta}_{i,j,k+\frac{1}{2}} = \frac{-\theta_{i,j,k+2} + 7\theta_{i,j,k+1} + 7\theta_{i,j,k} - \theta_{i,j,k-1}}{12} \tag{3.105}$$

3.2.4 Tracer advection

The tracer advection process is done after the time integration of the dynamical variables (ρ, ρu, ρv, ρw, and $\rho\theta$). We impose two constraints to tracer advection:

Consistency With Continuity (CWC) On the condition without any source/sink, the mass concentration in the advection process should be conserved along the trajectory. It is, at least, necessary that the spatially constant mass concentration should be kept in any motion of fluid. In order to satisfy this condition, we use the same mass flux at the last Runge-Kutta process of Eqs.() and () for integration of tracers:

$$\begin{aligned}\frac{(\rho q)^{n+1}_{i,j,k} - (\rho q)^{n}_{i,j,k}}{\Delta t} &= -\frac{(\rho u)_{i+\frac{1}{2},j,k}\overline{q}_{i+\frac{1}{2},j,k} - (\rho u)_{i-\frac{1}{2},j,k}\overline{q}_{i-\frac{1}{2},j,k}}{\Delta x} \\ &\quad -\frac{(\rho v)_{i,j+\frac{1}{2},k}\overline{q}_{i,j+\frac{1}{2},k} - (\rho v)_{i,j-\frac{1}{2},k}\overline{q}_{i,j-\frac{1}{2},k}}{\Delta y} \\ &\quad -\frac{(\rho w)_{i,j,k+\frac{1}{2}}\overline{q}_{i,j,k+\frac{1}{2}} - (\rho w)_{i,j,k-\frac{1}{2}}\overline{q}_{i,j,k-\frac{1}{2}}}{\Delta z}\end{aligned} \tag{3.106}$$

Monotonicity In order to satisfy the monotonicity of tracer advection, we employ the Flux Corrected Trasport scheme, which is a hybrid scheme with the 4th order central difference scheme and 1st order upwind scheme. If The 4th order central difference is applied, $\overline{q}$ is descritized as

$$\overline{q}^{high}_{i+\frac{1}{2},j,k} = \frac{-q_{i+2,j,k} + 7q_{i+1,j,k} + 7q_{i,j,k} - q_{i-1,j,k}}{12} \tag{3.107}$$

$$\overline{q}^{high}_{i,j+\frac{1}{2},k} = \frac{-q_{i,j+2,k} + 7q_{i,j+1,k} + 7q_{i,j,k} - q_{i,j-1,k}}{12} \tag{3.108}$$

$$\overline{q}^{high}_{i,j,k+\frac{1}{2}} = \frac{-q_{i,j,k+2} + 7q_{i,j,k+1} + 7q_{i,j,k} - q_{i,j,k-1}}{12}. \tag{3.109}$$

On the other hand, in the 1st order upwind scheme $\overline{q}$ is described as

$$\overline{q}^{low}_{i+\frac{1}{2},j,k} = \begin{cases} q_{i,j,k} & ((\rho u)_{i+\frac{1}{2},j,k} > 0) \\ q_{i+1,j,k} & (\text{otherwise}) \end{cases} \tag{3.110}$$

$$\overline{q}^{low}_{i,j+\frac{1}{2},k} = \begin{cases} q_{i,j,k} & ((\rho u)_{i,j+\frac{1}{2},k} > 0) \\ q_{i,j+1,k} & (\text{otherwise}) \end{cases} \tag{3.111}$$

$$\overline{q}^{low}_{i,j,k+\frac{1}{2}} = \begin{cases} q_{i,j,k} & ((\rho u)_{i,j,k+\frac{1}{2}} > 0) \\ q_{i,j,k+1} & (\text{otherwise}) \end{cases} \tag{3.112}$$

The actual $\overline{q}$ is described as

$$\overline{q}_{i+\frac{1}{2},j,k} = C_{i+\frac{1}{2},j,k}\overline{q}^{high}_{i+\frac{1}{2},j,k} + \left(1 - C_{i+\frac{1}{2},j,k}\right)\overline{q}^{low}_{i+\frac{1}{2},j,k} \quad (3.113)$$

$$\overline{q}_{i,j+\frac{1}{2},k} = C_{i,j+\frac{1}{2},k}\overline{q}^{high}_{i,j+\frac{1}{2},k} + \left(1 - C_{i,j+\frac{1}{2},k}\right)\overline{q}^{low}_{i,j+\frac{1}{2},k} \quad (3.114)$$

$$\overline{q}_{i,j,k+\frac{1}{2}} = C_{i,j,k+\frac{1}{2}}\overline{q}^{high}_{i,j,k+\frac{1}{2}} + \left(1 - C_{i,j,k+\frac{1}{2}}\right)\overline{q}^{low}_{i,j,k+\frac{1}{2}} \quad (3.115)$$

See the appedix for the method to determine the flux limter.

3.3 boundary condition

The boundary condition only for the vertical velocity at the top and bottom boundaries is needed:

$$w_{i,j,k_{max}+\frac{1}{2}} = 0 \quad (3.116)$$

$$w_{i,j,k_{min}-\frac{1}{2}} = 0 \quad (3.117)$$

This leads to the boundary condition of the prognostic variable as

$$(\rho w)_{i,j,k_{max}+\frac{1}{2}} = 0 \quad (3.118)$$

$$(\rho w)_{i,j,k_{min}-\frac{1}{2}} = 0 \quad (3.119)$$

3.4 Numerical filters

We impose an explicit numerical filter using the numerical viscosity and diffusion. Although the filter is necessary for numerical stability, too strong a filter could dampen any physically meaningful variability. In this subsection, we describe the numerical filters used in this model, and discuss the strength of the filter.

In order to damp the higher wavenumber component selectively, we adopt the hyperviscosity and diffusion in the traditional way. The hyperviscosity and diffusion of the nth order is defined as

$$\frac{\partial}{\partial x}\left[\nu\rho\frac{\partial^{n-1} f}{\partial x^{n-1}}\right], \quad (3.120)$$

where f is an arbitrary variable ($f \in \rho, u, v, w, \theta, q$).

The Laplacian of f is discretized as

$$\Delta f_i = \frac{1}{\Delta x_i}\left[\frac{1}{\Delta x_{i+\frac{1}{2}}} f_{i+1} - \left(\frac{1}{\Delta x_{i+\frac{1}{2}}} + \frac{1}{\Delta x_{i-\frac{1}{2}}}\right) f_i + \frac{1}{\Delta x_{i-\frac{1}{2}}} f_{i-1}\right], \quad (3.121)$$

and

$$\Delta^{n/2} f_i = \frac{1}{\Delta x_i}\left[\frac{1}{\Delta x_{i+\frac{1}{2}}}\Delta^{n/2-1} f_{i+1} - \left(\frac{1}{\Delta x_{i+\frac{1}{2}}} + \frac{1}{\Delta x_{i-\frac{1}{2}}}\right)\Delta^{n/2-1} f_i + \frac{1}{\Delta x_{i-\frac{1}{2}}}\Delta^{n/2-1} f_{i-1}\right]. \quad (3.122)$$

Here we consider spatially dependent grid interval in calculating the Laplacian. If it is calculated with constant Δx_i as

$$\Delta f_i = \frac{1}{\Delta x_i^2}\left(f_{i+1} - 2f_i + f_{i-1}\right), \tag{3.123}$$

$$\Delta^{n/2} f_i = \frac{1}{\Delta x_i^2}\left(\Delta^{n/2-1} f_{i+1} - 2\Delta^{n/2-1} f_i + \Delta^{n/2-1} f_{i-1}\right), \tag{3.124}$$

non-negligible numerical noise appears where the grid spacing varies (e.g., stretching layer near the top boundary).

The hyperviscosity and diffusion can be discretized as

$$\frac{\partial}{\partial x}\left[\nu\rho\frac{\partial^{n-1} f}{\partial^{n-1} x}\right] \sim \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x_i}, \tag{3.125}$$

where

$$F_{i+\frac{1}{2}} \frac{\nu_{i+\frac{1}{2}}\rho_{i+\frac{1}{2}}}{\Delta x_{i+\frac{1}{2}}}\left(\Delta^{n/2-1} f_{i+1} - \Delta^{n/2-1} f_i\right). \tag{3.126}$$

The coefficient, ν, is written as

$$\nu_{i+\frac{1}{2}} = (-1)^{n/2+1}\gamma\frac{\Delta x^n_{i+\frac{1}{2}}}{2^n \Delta t}, \tag{3.127}$$

where γ is a non-dimensional coefficient. One-dimensional sinusoidal two-grid noise will decay to $1/e$ with $1/\gamma$ time steps. Note that the theoretical e-folding time is $\frac{2^n}{\pi^n}\frac{\Delta t}{\gamma}$. However, it is $\frac{\Delta t}{\gamma}$ with the fourth-order central scheme used in this model.

For the numerical stability of the numerical filter itself, it should satisfy

$$\gamma < 1 \tag{3.128}$$

for the one-dimensional two-grid noise, and

$$\gamma < \frac{1}{3} \tag{3.129}$$

for the three-dimensional two-grid noise. The conditions might be stricter for other types of noise.

The flux, F, for the numerical filter is added to the advective flux as

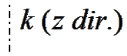
$$(\rho u f)^{\dagger}_{i+\frac{1}{2}} = (\rho u f)_{i+\frac{1}{2}} + F_{i+\frac{1}{2}}, \tag{3.130}$$

where the first term of the right-hand side is the flux calculated by the advection scheme. In the present model, the advection scheme is the fourth-order central difference scheme. This concept is very important for the CWC condition in the tracer equations. The modified mass flux of the numerical filter should be used in the tracer advection, otherwise the CWC condition is violated.

The numerical viscosity and diffusion in the y and z directions are formulated in the same way as in the x direction, although a special treatment for the z direction is needed. At the top and bottom boundaries, the flux must be zero, $F_{k_{\max}+\frac{1}{2}} = F_{k_{\min}-\frac{1}{2}} = 0$. In order to calculate the $F_{k_{\max}-\frac{1}{2}}$ and $F_{k_{\min}+\frac{1}{2}}$, values beyond the boundaries, $f_{k_{\mathrm{max}}+1}$ and $f_{k_{\mathrm{min}}-1}$, are required, then the mirror boundary condition is assumed; $f_{k_{\mathrm{max}}+1} = -f_{k_{\mathrm{max}}}$ and $f_{k_{\mathrm{min}}-1} = -f_{k_{\mathrm{min}}}$. This condition is appropriate to cause the decay the vertical two-grid noise.

Vertical profiles of density, potential temperature, and water vapor usually have significant (e.g., logarithmic) dependencies on height. Eq. (3.125) has a non-zero value even for the steady state, and the numerical filter produces artificial motion. To reduce this artificial motion, we introduce a reference profile which is a function of height, and deviation from the reference is used as f instead of ρ, θ, and q_{v} in calculating the numerical filter. The reference profile can be chosen arbitrarily, but a profile under hydrostatic balance is usually chosen.

(a) Control volume for the mass



(b) Control volume for the momentum

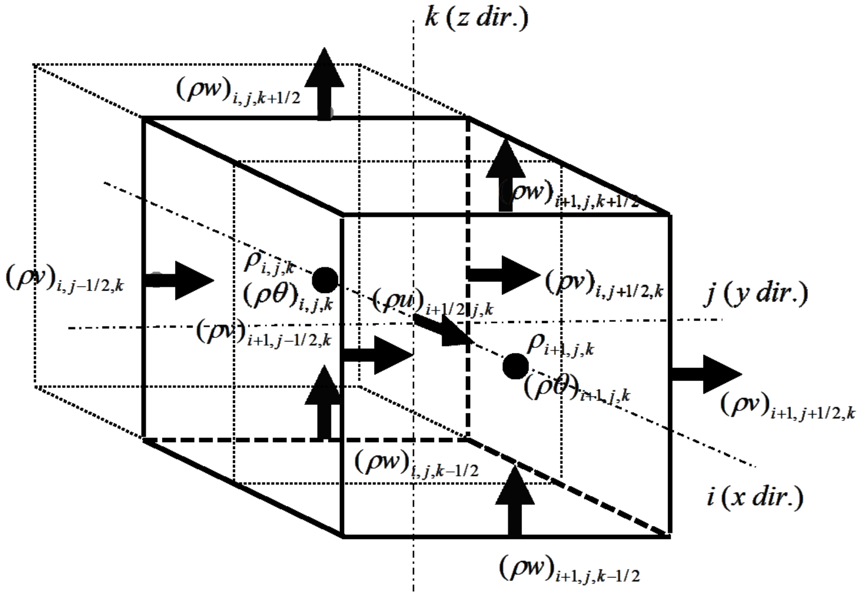


Figure 3.1: Control volume

Chapter 4

Terrain-following Coordinates

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4.1 Geometry and Definitions

We introduce a terrain following coordinate system with a new vertical coordinate ξ. ξ-coordinate system is not deformable system. We use the relation between z and ξ as

$$\xi = \frac{z_{toa}(z - z_{sfc})}{z_{toa} - z_{sfc}}, \tag{4.1}$$

Where z_{toa} is the top of the model domain and z_{sfc} is the surface height, which depends on the horizontal location.

The metrics are defined as

$$G^{\frac{1}{2}} = \frac{\partial z}{\partial \xi}, \tag{4.2}$$

$$J_{13}^{\xi} = \left(\frac{\partial \xi}{\partial x}\right)_z = -\frac{J_{13}^z}{J_{33}^z}, \tag{4.3}$$

$$J_{23}^{\xi} = \left(\frac{\partial \xi}{\partial y}\right)_z = -\frac{J_{23}^z}{J_{33}^z}, \tag{4.4}$$

$$J_{33}^{\xi} = \frac{\partial \xi}{\partial z} = \frac{1}{J_{33}^z}, \tag{4.5}$$

where

$$J_{13}^z = \left(\frac{\partial z}{\partial x}\right)_\xi, \tag{4.6}$$

$$J_{23}^z = \left(\frac{\partial z}{\partial y}\right)_\xi, \tag{4.7}$$

$$J_{33}^z = G^{\frac{1}{2}} \tag{4.8}$$

If we use the Eqs.(4.1)-(4.5), we obtain following equations:

$$G^{\frac{1}{2}}\nabla\phi = \left[\left(\frac{\partial G^{\frac{1}{2}}\phi}{\partial x}\right)_\xi + \frac{\partial J^\xi_{13}G^{\frac{1}{2}}\phi}{\partial \xi}\right]\hat{\boldsymbol{e}}_x + \left[\left(\frac{\partial G^{\frac{1}{2}}\phi}{\partial y}\right)_\xi + \frac{\partial J^\xi_{23}G^{\frac{1}{2}}\phi}{\partial \xi}\right]\hat{\boldsymbol{e}}_y + \left[\frac{\partial J^\xi_{33}G^{\frac{1}{2}}\phi}{\partial \xi}\right]\hat{\boldsymbol{e}}_z, \tag{4.9}$$

$$G^{\frac{1}{2}}\nabla\cdot(\phi\mathbf{u}) = \left(\frac{\partial G^{\frac{1}{2}}\phi u}{\partial x}\right)_\xi + \left(\frac{\partial G^{\frac{1}{2}}\phi v}{\partial y}\right)_\xi + \frac{\partial G^{\frac{1}{2}}\phi\dot{\xi}}{\partial \xi} \tag{4.10}$$

where $\{\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z\}$ are unit vectors in Cartesian coordinate, and $\dot{\xi}$ is the vertical velocity component in the terrain following coordinate, giving by

$$\dot{\xi} \equiv \frac{d\xi}{dt} = J^\xi_{13}u + J^\xi_{23}v + J^\xi_{33}w. \tag{4.11}$$

4.2 Summary of modified equations in the dynamical process

Prognostic variables by multiplying $G^{\frac{1}{2}}$ are defined as

$$(\rho Q_v)_{i,j,k} = G^{\frac{1}{2}}_{i,j,k}(\rho q_v)_{i,j,k}, \tag{4.12}$$

$$(\rho Q_l)_{i,j,k} = G^{\frac{1}{2}}_{i,j,k}(\rho q_l)_{i,j,k}, \tag{4.13}$$

$$(\rho Q_s)_{i,j,k} = G^{\frac{1}{2}}_{i,j,k}(\rho q_s)_{i,j,k}, \tag{4.14}$$

$$R_{i,j,k} = G^{\frac{1}{2}}_{i,j,k}\rho_{i,j,k}, \tag{4.15}$$

$$(\rho U)_{i+\frac{1}{2},j,k} = G^{\frac{1}{2}}_{i+\frac{1}{2},j,k}(\rho u)_{i+\frac{1}{2},j,k}, \tag{4.16}$$

$$(\rho V)_{i,j+\frac{1}{2},k} = G^{\frac{1}{2}}_{i,j+\frac{1}{2},k}(\rho v)_{i,j+\frac{1}{2},k}, \tag{4.17}$$

$$(\rho W)_{i,j,k+\frac{1}{2}} = G^{\frac{1}{2}}_{i,j,k+\frac{1}{2}}(\rho w)_{i,j,k+\frac{1}{2}}, \tag{4.18}$$

$$(\rho \Theta)_{i,j,k} = G^{\frac{1}{2}}_{i,j,k}(\rho\theta)_{i,j,k}, \tag{4.19}$$

$$P_{i,j,k} = G^{\frac{1}{2}}_{i,j,k}p_{i,j,k}. \tag{4.20}$$

Eqs.(2.67)-(2.72) are modified using Eqs.(4.9)-(4.11),

$$\frac{\partial \rho Q_v}{\partial t} + \frac{\partial}{\partial x_j}(\rho Q_v u_j) = 0, \tag{4.21}$$

$$\frac{\partial \rho Q_l}{\partial t} + \frac{\partial}{\partial x_j}(\rho Q_l u_j) = 0, \tag{4.22}$$

$$\frac{\partial \rho Q_s}{\partial t} + \frac{\partial}{\partial x_j}(\rho Q_s u_j) = 0, \tag{4.23}$$

$$\frac{\partial R}{\partial t} + \frac{\partial}{\partial x_j}(R u_j) = 0, \tag{4.24}$$

$$\frac{\partial \rho U}{\partial t} + \frac{\partial}{\partial x_j}(\rho U u_j) = -\left(\frac{\partial P}{\partial x}\right)_\xi - \frac{\partial J_{13}^\xi P}{\partial \xi}, \tag{4.25}$$

$$\frac{\partial \rho V}{\partial t} + \frac{\partial}{\partial x_j}(\rho V u_j) = -\left(\frac{\partial P}{\partial y}\right)_\xi - \frac{\partial J_{23}^\xi P}{\partial \xi}, \tag{4.26}$$

$$\frac{\partial \rho W}{\partial t} + \frac{\partial}{\partial x_j}(\rho W u_j) = -\frac{\partial J_{33}^\xi P}{\partial \xi} - Rg, \tag{4.27}$$

$$\frac{\partial \rho \Theta}{\partial t} + \frac{\partial}{\partial x_j}(\rho \Theta u_j) = 0, \tag{4.28}$$

where Einstein summation has been used to implicitly sum over repeated indices, and $(x_1, x_2, x_3) = (x, y, \xi)$, $(u_1, u_2, u_3) = (u, v, \dot{\xi})$.

4.3 Spatial descretization

4.3.1 Continuity equation

$$\begin{aligned}
\left(\frac{\partial R}{\partial t}\right)_{i,j,k} = -\Bigg[& \frac{(\rho U)_{i+\frac{1}{2},j,k} - (\rho U)_{i-\frac{1}{2},j,k}}{\Delta x} \\
& + \frac{(\rho V)_{i,j+\frac{1}{2},k} - (\rho V)_{i,j-\frac{1}{2},k}}{\Delta y} \\
& + \frac{(J_{13}^\xi)_{i,j,k+\frac{1}{2}} \overline{\widetilde{(\rho U)}^{x}}^{z}_{i,j,k+\frac{1}{2}} - (J_{13}^\xi)_{i,j,k-\frac{1}{2}} \overline{\widetilde{(\rho U)}^{x}}^{z}_{i,j,k-\frac{1}{2}}}{\Delta \xi} \\
& + \frac{(J_{23}^\xi)_{i,j,k+\frac{1}{2}} \overline{\widetilde{(\rho V)}^{y}}^{z}_{i,j,k+\frac{1}{2}} - (J_{23}^\xi)_{i,j,k-\frac{1}{2}} \overline{\widetilde{(\rho V)}^{y}}^{z}_{i,j,k-\frac{1}{2}}}{\Delta \xi} \\
& + \frac{(J_{33}^\xi)_{i,j,k+\frac{1}{2}} (\rho W)_{i,j,k+\frac{1}{2}} - (J_{33}^\xi)_{i,j,k+\frac{1}{2}} (\rho W)_{i,j,k-\frac{1}{2}}}{\Delta \xi} \Bigg]
\end{aligned} \tag{4.29}$$

where

$$\overline{\widetilde{(\rho U)}^{x}}^{z}_{i,j,k+\frac{1}{2}} = G^{\frac{1}{2}}_{i,j,k+\frac{1}{2}} \frac{\widetilde{(\rho u)}^{x}_{i,j,k+1} + \widetilde{(\rho u)}^{x}_{i,j,k}}{2}, \tag{4.30}$$

$$\overline{\widetilde{(\rho V)}^{y}}^{z}_{i,j,k+\frac{1}{2}} = G^{\frac{1}{2}}_{i,j,k+\frac{1}{2}} \frac{\widetilde{(\rho v)}^{y}_{i,j,k+1} + \widetilde{(\rho v)}^{y}_{i,j,k}}{2}, \tag{4.31}$$

$\widetilde{(\rho u)}^{x}_{i,j,k}$ and $\widetilde{(\rho v)}^{y}_{i,j,k}$ are obtained by same manner in eq.(3.20)

4.3.2 Momentum equations

$$
\begin{aligned}
\left(\frac{\partial \rho U}{\partial t}\right)_{i+\frac{1}{2},j,k} = -\Bigg[& \frac{\widetilde{(\rho U)}^{x}_{i+1,j,k}\overline{u}_{i+1,j,k} - \widetilde{(\rho U)}^{x}_{i,j,k}\overline{u}_{i,j,k}}{\Delta x} \\
& + \frac{\widetilde{(\rho U)}^{y}_{i+\frac{1}{2},j+\frac{1}{2},k}\overline{v}_{i+\frac{1}{2},j+\frac{1}{2},k} - \widetilde{(\rho U)}^{y}_{i+\frac{1}{2},j-\frac{1}{2},k}\overline{v}_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta y} \\
& + \frac{(J^{\xi}_{13})_{i+\frac{1}{2},j,k+\frac{1}{2}}\widetilde{(\rho U)}^{z}_{i+\frac{1}{2},j,k+\frac{1}{2}}\overline{\overline{u}}^{z}_{i+\frac{1}{2},j,k+\frac{1}{2}} - (J^{\xi}_{13})_{i+\frac{1}{2},j,k-\frac{1}{2}}\widetilde{(\rho U)}^{z}_{i+\frac{1}{2},j,k-\frac{1}{2}}\overline{\overline{u}}^{z}_{i+\frac{1}{2},j,k-\frac{1}{2}}}{\Delta \xi} \\
& + \frac{(J^{\xi}_{23})_{i+\frac{1}{2},j,k+\frac{1}{2}}\widetilde{(\rho U)}^{z}_{i+\frac{1}{2},j,k+\frac{1}{2}}\overline{\overline{v}^{y}}^{xz}_{i+\frac{1}{2},j,k+\frac{1}{2}} - (J^{\xi}_{23})_{i+\frac{1}{2},j,k-\frac{1}{2}}\widetilde{(\rho U)}^{z}_{i+\frac{1}{2},j,k-\frac{1}{2}}\overline{\overline{v}^{y}}^{xz}_{i+\frac{1}{2},j,k-\frac{1}{2}}}{\Delta \xi} \\
& + \frac{(J^{\xi}_{33})_{i+\frac{1}{2},j,k+\frac{1}{2}}\widetilde{(\rho U)}^{z}_{i+\frac{1}{2},j,k+\frac{1}{2}}\overline{\overline{w}}^{x}_{i+\frac{1}{2},j,k+\frac{1}{2}} - (J^{\xi}_{33})_{i+\frac{1}{2},j,k-\frac{1}{2}}\widetilde{(\rho U)}^{z}_{i+\frac{1}{2},j,k-\frac{1}{2}}\overline{\overline{w}}^{x}_{i+\frac{1}{2},j,k-\frac{1}{2}}}{\Delta \xi} \\
& + \frac{P_{i+1,j,k} - P_{i,j,k}}{\Delta x} \\
& + \frac{(J^{\xi}_{13})_{i+\frac{1}{2},j,k+\frac{1}{2}}\overline{P}^{xz}_{i+\frac{1}{2},j,k+\frac{1}{2}} - (J^{\xi}_{13})_{i+\frac{1}{2},j,k-\frac{1}{2}}\overline{P}^{xz}_{i+\frac{1}{2},j,k-\frac{1}{2}}}{\Delta \xi}\Bigg],
\end{aligned}
\tag{4.32}
$$

where $\widetilde{(\rho U)}^{x}_{i,j,k}$, $\widetilde{(\rho U)}^{y}_{i+\frac{1}{2},j+\frac{1}{2},k}$ and $\widetilde{(\rho U)}^{z}_{i+\frac{1}{2},j,k+\frac{1}{2}}$ is obtained according to the method of eq(3.20)-(3.22). The velocities at the cell wall for the staggered control volume to x direction are defined by eq(3.23)-(3.25). $\overline{\overline{u}}^{z}$ and $\overline{\overline{v}^{y}}^{xz}$ are defined as

$$
\overline{\overline{u}}^{z}_{i+\frac{1}{2},j,k+\frac{1}{2}} = \frac{\overline{u}_{i+\frac{1}{2},j,k+1} + \overline{u}_{i+\frac{1}{2},j,k}}{2}, \tag{4.33}
$$

$$
\overline{\overline{v}^{y}}^{xz}_{i+\frac{1}{2},j,k+\frac{1}{2}} = \frac{\overline{v}^{y}_{i+1,j,k+1} + \overline{v}^{y}_{i+1,j,k} + \overline{v}^{y}_{i,j,k+1} + \overline{v}^{y}_{i,j,k}}{4}. \tag{4.34}
$$

$\overline{P}^{xz}$ is defined as

$$
\overline{P}^{xz}_{i+\frac{1}{2},j,k+\frac{1}{2}} = G^{\frac{1}{2}}_{i+\frac{1}{2},j,k+\frac{1}{2}}\frac{p_{i+1,j,k+1} + p_{i+1,j,k} + p_{i,j,k+1} + p_{i,j,k}}{4}. \tag{4.35}
$$

The momentum equations in the y and z directions are descretized in the same way:

$$
\begin{aligned}
\left(\frac{\partial \rho V}{\partial t}\right)_{i,j+\frac{1}{2},k} = -\Bigg[& \frac{\widetilde{(\rho V)}^{x}_{i+\frac{1}{2},j+\frac{1}{2},k}\overline{u}_{i+\frac{1}{2},j+\frac{1}{2},k} - \widetilde{(\rho V)}^{x}_{i-\frac{1}{2},j+\frac{1}{2},k}\overline{u}_{i-\frac{1}{2},j+\frac{1}{2},k}}{\Delta x} \\
& + \frac{\widetilde{(\rho V)}^{y}_{i,j+1,k}\overline{v}_{i,j+1,k} - \widetilde{(\rho V)}^{y}_{i,j,k}\overline{v}_{i,j,k}}{\Delta y} \\
& + \frac{(J^{\xi}_{13})_{i,j+\frac{1}{2},k+\frac{1}{2}}\widetilde{(\rho V)}^{z}_{i,j+\frac{1}{2},k+\frac{1}{2}}\overline{\overline{u}^{x}}^{yz}_{i,j+\frac{1}{2},k+\frac{1}{2}} - (J^{\xi}_{13})_{i,j+\frac{1}{2},k-\frac{1}{2}}\widetilde{(\rho V)}^{z}_{i,j+\frac{1}{2},k-\frac{1}{2}}\overline{\overline{u}^{x}}^{yz}_{i,j+\frac{1}{2},k-\frac{1}{2}}}{\Delta \xi} \\
& + \frac{(J^{\xi}_{23})_{i,j+\frac{1}{2},k+\frac{1}{2}}\widetilde{(\rho V)}^{z}_{i,j+\frac{1}{2},k+\frac{1}{2}}\overline{\overline{v}}^{z}_{i,j+\frac{1}{2},k+\frac{1}{2}} - (J^{\xi}_{23})_{i,j+\frac{1}{2},k-\frac{1}{2}}\widetilde{(\rho V)}^{z}_{i,j+\frac{1}{2},k-\frac{1}{2}}\overline{\overline{v}}^{z}_{i,j+\frac{1}{2},k-\frac{1}{2}}}{\Delta \xi} \\
& + \frac{(J^{\xi}_{33})_{i,j+\frac{1}{2},k+\frac{1}{2}}\widetilde{(\rho V)}^{z}_{i,j+\frac{1}{2},k+\frac{1}{2}}\overline{\overline{w}}^{y}_{i,j+\frac{1}{2},k+\frac{1}{2}} - (J^{\xi}_{33})_{i,j+\frac{1}{2},k-\frac{1}{2}}\widetilde{(\rho V)}^{z}_{i,j+\frac{1}{2},k-\frac{1}{2}}\overline{\overline{w}}^{y}_{i,j+\frac{1}{2},k-\frac{1}{2}}}{\Delta \xi} \\
& + \frac{P_{i,j+1,k} - P_{i,j,k}}{\Delta y} \\
& + \frac{(J^{\xi}_{23})_{i,j+\frac{1}{2},k+\frac{1}{2}}\overline{P}^{yz}_{i,j+\frac{1}{2},k+\frac{1}{2}} - (J^{\xi}_{23})_{i,j+\frac{1}{2},k-\frac{1}{2}}\overline{P}^{yz}_{i,j+\frac{1}{2},k-\frac{1}{2}}}{\Delta \xi}\Bigg],
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
\left(\frac{\partial \rho W}{\partial t}\right)_{i,j,k+\frac{1}{2}} = -\Bigg[& \frac{\widetilde{(\rho W)}^{x}_{i+\frac{1}{2},j,k+\frac{1}{2}}\overline{u}_{i+\frac{1}{2},j,k+\frac{1}{2}} - \widetilde{(\rho W)}^{x}_{i-\frac{1}{2},j,k+\frac{1}{2}}\overline{u}_{i-\frac{1}{2},j,k+\frac{1}{2}}}{\Delta x} \\
& + \frac{\widetilde{(\rho W)}^{y}_{i,j+\frac{1}{2},k+\frac{1}{2}}\overline{v}_{i,j+\frac{1}{2},k+\frac{1}{2}} - \widetilde{(\rho W)}^{y}_{i,j-\frac{1}{2},k+\frac{1}{2}}\overline{v}_{i,j-\frac{1}{2},k+\frac{1}{2}}}{\Delta y} \\
& + \frac{(J^{\xi}_{13})_{i,j,k+1}\widetilde{(\rho W)}^{z}_{i,j,k+1}\overline{u}^{x}_{i,j,k+1} - (J^{\xi}_{13})_{i,j,k}\widetilde{(\rho W)}^{z}_{i,j,k}\overline{u}^{x}_{i,j,k}}{\Delta \xi} \\
& + \frac{(J^{\xi}_{23})_{i,j,k+1}\widetilde{(\rho W)}^{z}_{i,j,k+1}\overline{v}^{y}_{i,j,k+1} - (J^{\xi}_{23})_{i,j,k}\widetilde{(\rho W)}^{z}_{i,j,k}\overline{v}^{y}_{i,j,k}}{\Delta \xi} \\
& + \frac{(J^{\xi}_{33})_{i,j,k+1}\widetilde{(\rho W)}^{z}_{i,j,k+1}\overline{w}^{z}_{i,j,k+1} - (J^{\xi}_{33})_{i,j,k}\widetilde{(\rho W)}^{z}_{i,j,k}\overline{w}^{z}_{i,j,k}}{\Delta \xi} \\
& + \frac{(J^{\xi}_{33})_{i,j,k+1}P_{i,j,k+1} - (J^{\xi}_{33})_{i,j,k}P_{i,j,k}}{\Delta \xi}\Bigg].
\end{aligned}
\tag{4.37}
$$

4.3.3 Energy equation

$$
\begin{aligned}
\left(\frac{\partial \rho \Theta}{\partial t}\right)_{i,j,k} = -\Bigg[& \frac{(\rho U)_{i+\frac{1}{2},j,k}\overline{\theta}_{i+\frac{1}{2},j,k} - (\rho U)_{i-\frac{1}{2},j,k}\overline{\theta}_{i-\frac{1}{2},j,k}}{\Delta x} \\
& + \frac{(\rho V)_{i,j+\frac{1}{2},k}\overline{\theta}_{i,j+\frac{1}{2},k} - (\rho V)_{i,j-\frac{1}{2},k}\overline{\theta}_{i,j-\frac{1}{2},k}}{\Delta y} \\
& + \frac{(J^{\xi}_{13})_{i,j,k+\frac{1}{2}}\overline{\widetilde{(\rho U)}^{x}}^{z}_{i,j,k+\frac{1}{2}}\overline{\theta}_{i,j,k+\frac{1}{2}} - (J^{\xi}_{13})_{i,j,k-\frac{1}{2}}\overline{\widetilde{(\rho U)}^{x}}^{z}_{i,j,k-\frac{1}{2}}\overline{\theta}_{i,j,k-\frac{1}{2}}}{\Delta \xi} \\
& + \frac{(J^{\xi}_{23})_{i,j,k+\frac{1}{2}}\overline{\widetilde{(\rho V)}^{y}}^{z}_{i,j,k+\frac{1}{2}}\overline{\theta}_{i,j,k+\frac{1}{2}} - (J^{\xi}_{23})_{i,j,k-\frac{1}{2}}\overline{\widetilde{(\rho V)}^{y}}^{z}_{i,j,k-\frac{1}{2}}\overline{\theta}_{i,j,k-\frac{1}{2}}}{\Delta \xi} \\
& + \frac{(J^{\xi}_{33})_{i,j,k+\frac{1}{2}}(\rho W)_{i,j,k+\frac{1}{2}}\overline{\theta}_{i,j,k+\frac{1}{2}} - (J^{\xi}_{33})_{i,j,k+\frac{1}{2}}(\rho W)_{i,j,k-\frac{1}{2}}\overline{\theta}_{i,j,k-\frac{1}{2}}}{\Delta \xi} \Bigg]
\end{aligned}
\tag{4.38}
$$

where $\overline{\theta}_{i+\frac{1}{2},j,k}$, $\overline{\theta}_{i,j+\frac{1}{2},k}$ and $\overline{\theta}_{i,j,k+\frac{1}{2}}$ are obtained according to the method of eq(3.29)-(3.31).

Chapter 5

Map factor

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5.1 Coordinate transform

A orthogonal rectangular coordinate (x, y, z). A orthogonal curvilinear coordinate (ξ, η, ζ).

The transform is defined by

$$\boldsymbol{e}_\xi = \frac{\partial x}{\partial \xi}\hat{\boldsymbol{e}}_x + \frac{\partial y}{\partial \xi}\hat{\boldsymbol{e}}_y + \frac{\partial z}{\partial \xi}\hat{\boldsymbol{e}}_z, \tag{5.1}$$

$$\boldsymbol{e}_\eta = \frac{\partial x}{\partial \eta}\hat{\boldsymbol{e}}_x + \frac{\partial y}{\partial \eta}\hat{\boldsymbol{e}}_y + \frac{\partial z}{\partial \eta}\hat{\boldsymbol{e}}_z, \tag{5.2}$$

$$\boldsymbol{e}_\zeta = \frac{\partial x}{\partial \zeta}\hat{\boldsymbol{e}}_x + \frac{\partial y}{\partial \zeta}\hat{\boldsymbol{e}}_y + \frac{\partial z}{\partial \zeta}\hat{\boldsymbol{e}}_z. \tag{5.3}$$

Reverse transform is

$$\hat{\boldsymbol{e}}_x = \frac{\partial \xi}{\partial x}\boldsymbol{e}_\xi + \frac{\partial \eta}{\partial x}\boldsymbol{e}_\eta + \frac{\partial \zeta}{\partial x}\boldsymbol{e}_\zeta, \tag{5.4}$$

$$\hat{\boldsymbol{e}}_y = \frac{\partial \eta}{\partial y}\boldsymbol{e}_\xi + \frac{\partial \eta}{\partial y}\boldsymbol{e}_\eta + \frac{\partial \zeta}{\partial y}\boldsymbol{e}_\zeta, \tag{5.5}$$

$$\hat{\boldsymbol{e}}_z = \frac{\partial \zeta}{\partial z}\boldsymbol{e}_\xi + \frac{\partial \eta}{\partial z}\boldsymbol{e}_\eta + \frac{\partial \zeta}{\partial z}\boldsymbol{e}_\zeta. \tag{5.6}$$

The Jacobian matrix is $\{\frac{\partial \xi^k}{\partial x^i}\}$.

The reverse transform after the transform of the transform after the reverse transform make a vector to the original vector;

$$\frac{\partial \xi^k}{\partial x^i}\frac{\partial x^i}{\partial \xi^l} = \delta^k_l, \tag{5.7}$$

$$\frac{\partial x^i}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^j} = \delta^i_j, \tag{5.8}$$

where index which appares upper and lower suffix in a single term implies summation of the term over set $1, 2, 3$ (Einstein notation).

Spatial parial derivative is tranformed with the Jacobian matrix (covariant transform);

$$\frac{\partial}{\partial \xi^k} = \frac{\partial x^i}{\partial \xi^k}\frac{\partial}{\partial x^i}, \tag{5.9}$$

$$\frac{\partial}{\partial x^i} = \frac{\partial \xi^k}{\partial x^i}\frac{\partial}{\partial \xi^k}. \tag{5.10}$$

Velocity is transformed with the inverse of the Jacobian matrix (cotravariant transform);

$$d\xi^k = \frac{\partial \xi^k}{\partial x^i}dx^i, \tag{5.11}$$

$$dx^i = \frac{\partial x^i}{\partial \xi^k}d\xi^k. \tag{5.12}$$

The metric tensor, g_{kl} is defined by

$$\begin{aligned} g_{kl} = \boldsymbol{e}_k \cdot \boldsymbol{e}_l &= \left(\frac{\partial x^i}{\partial \xi_k}\hat{\boldsymbol{e}}_i\right)\cdot\left(\frac{\partial x^j}{\partial \xi_l}\hat{\boldsymbol{e}}_j\right) \\ &= \frac{\partial x^i}{\partial \xi_k}\frac{\partial x^i}{\partial \xi_l}. \end{aligned} \tag{5.13}$$

For orthogonal curvilinear coordinates, the matrix g_{kl} is diagonal. Metric factor, h_k is defined as

$$h_k^2 = g_{kk} = \sum_i \left(\frac{\partial x^i}{\partial \xi_k}\right)^2. \tag{5.14}$$

Here we define the matrix, $\boldsymbol{E}_\xi$ is

$$\boldsymbol{E}_\xi = (\boldsymbol{e}_\xi\ \boldsymbol{e}_\eta\ \boldsymbol{e}_\zeta)\cdot\boldsymbol{H}^{-1} = \boldsymbol{E}_x \cdot \left\{\frac{\partial x^i}{\partial \xi^k}\right\}\cdot\boldsymbol{H}^{-1}, \tag{5.15}$$

where $\boldsymbol{E}_x = (\hat{\boldsymbol{e}}_x\ \hat{\boldsymbol{e}}_y\ \hat{\boldsymbol{e}}_z)$, and

$$\boldsymbol{H} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}. \tag{5.16}$$

The vector $\frac{1}{h_k}\boldsymbol{e}_k$ is unit vector and orthogonal each other, so the inverse of the $\boldsymbol{E}_\xi$ is $\boldsymbol{E}_\xi^T$.

$$\left(\boldsymbol{E}_x \cdot \left\{\frac{\partial x^i}{\partial \xi^k}\right\}\cdot\boldsymbol{H}^{-1}\right)^{-1} = \left(\boldsymbol{E}_x \cdot \left\{\frac{\partial x^i}{\partial \xi^k}\right\}\cdot\boldsymbol{H}^{-1}\right)^T,$$

$$\boldsymbol{H}\cdot\left\{\frac{\partial \xi^i}{\partial x^k}\right\}\cdot\boldsymbol{E}_x^{-1} = \boldsymbol{H}^{-1}\cdot\left\{\frac{\partial x^i}{\partial \xi^k}\right\}^T\cdot\boldsymbol{E}_x^T,$$

$$\begin{aligned} \left\{\frac{\partial \xi^i}{\partial x^k}\right\} &= \boldsymbol{H}^{-2}\cdot\left\{\frac{\partial x^i}{\partial \xi^k}\right\}^T\cdot\boldsymbol{E}_x^T\cdot\boldsymbol{E}_x \\ &= \boldsymbol{H}^{-2}\cdot\left\{\frac{\partial x^k}{\partial \xi^i}\right\}. \end{aligned} \tag{5.17}$$

That is

$$\frac{\partial \xi^k}{\partial x^i} = \frac{1}{h_k^2} \frac{\partial x^i}{\partial \xi^k}. \tag{5.18}$$

5.2 Governing equations

5.2.1 Continuous equiation

Divergence of $\rho \boldsymbol{u}$ is

$$\begin{aligned}
\frac{\partial}{\partial x^i}(\rho dx^i) &= \frac{\partial \xi^k}{\partial x^i} \frac{\partial}{\partial \xi^k} \left(\rho \frac{\partial x^i}{\partial \xi^l} d\xi^l \right) \\
&= \frac{\partial \xi^k}{\partial x^i} \frac{\partial x^i}{\partial \xi^l} \frac{\partial}{\partial \xi^k}(\rho d\xi^l) + \rho d\xi^l \frac{\partial \xi^k}{\partial x^i} \frac{\partial^2 x^i}{\partial \xi^k \partial \xi^l} \\
&= \frac{\partial}{\partial \xi^k}(\rho d\xi^k) + \sum_k \frac{1}{h_k^2} \rho d\xi^l \frac{\partial x^i}{\partial \xi^k} \frac{\partial^2 x^i}{\partial \xi^k \partial \xi^l} \\
&= \frac{\partial}{\partial \xi^k}(\rho d\xi^k) + \sum_k \frac{1}{2h_k^2} \rho d\xi^l \frac{\partial}{\partial \xi^l} \left(\frac{\partial x^i}{\partial \xi^k} \right)^2 \\
&= \frac{\partial}{\partial \xi^k}(\rho d\xi^k) + \sum_k \frac{1}{2h_k^2} \rho d\xi^l \frac{\partial}{\partial \xi^l} h_k^2 \\
&= \frac{\partial}{\partial \xi^k}(\rho d\xi^k) + \sum_k \frac{1}{2} \rho d\xi^l \frac{\partial}{\partial \xi^l} \ln h_k^2 \\
&= \frac{\partial}{\partial \xi^k}(\rho d\xi^k) + \rho d\xi^k \frac{\partial}{\partial \xi^k} \ln(\prod_l h_l) \\
&= J \left\{ J^{-1} \frac{\partial}{\partial \xi^k}(\rho d\xi^l) + \rho d\xi^k \frac{\partial}{\partial \xi^k} J^{-1} \right\} \\
&= J \frac{\partial}{\partial \xi^k}(J^{-1} \rho d\xi^k),
\end{aligned} \tag{5.19}$$

where J is the Jacobian of the Jacobian matrix and

$$J = \frac{1}{\prod_k h_k}. \tag{5.20}$$

The continuous equation is

$$\frac{\partial \rho}{\partial t} + J \frac{\partial}{\partial \xi^k} \frac{\rho d\xi^k}{J} = 0. \tag{5.21}$$

5.2.2 Momentum equation

$$\frac{\partial \rho d\xi^k}{\partial t} = \frac{\partial \xi^k}{\partial x^i} \frac{\partial \rho dx^i}{\partial t}. \tag{5.22}$$

Advection term

$$
\begin{aligned}
&\frac{\partial \xi^k}{\partial x^i}\frac{\partial \rho dx^i dx^j}{\partial x^j}\\
&= \frac{\partial \xi^k}{\partial x^i}\left(dx^j \frac{\partial \rho dx^i}{\partial x^j} + \rho dx^i \frac{\partial dx^j}{\partial x^j}\right)\\
&= \frac{\partial \xi^k}{\partial x^i}\left(\frac{\partial x^j}{\partial \xi^l}d\xi^l\right)\frac{\partial \xi^m}{\partial x^j}\frac{\partial}{\partial \xi^m}\left(\rho\frac{\partial x^i}{\partial \xi^n}d\xi^n\right) + \frac{\partial \xi^k}{\partial x^i}\rho\left(\frac{\partial x^i}{\partial \xi^l}d\xi^l\right)\frac{\partial \xi^m}{\partial x^j}\frac{\partial}{\partial \xi^m}\left(\frac{\partial x^j}{\partial \xi^n}d\xi^n\right)\\
&= d\xi^l\frac{\partial \xi^k}{\partial x^i}\frac{\partial}{\partial \xi^l}\left(\rho\frac{\partial x^i}{\partial \xi^n}d\xi^n\right) + \rho d\xi^k\frac{\partial \xi^m}{\partial x^j}\frac{\partial}{\partial \xi^m}\left(\frac{\partial x^j}{\partial \xi^n}d\xi^n\right)\\
&= d\xi^l\frac{\partial \xi^k}{\partial x^i}\left\{\frac{\partial x^i}{\partial \xi^n}\frac{\partial}{\partial \xi^l}\left(\rho d\xi^n\right) + \rho d\xi^n\frac{\partial^2 x^i}{\partial \xi^l \partial \xi^n}\right\} + \rho d\xi^k\frac{\partial \xi^m}{\partial x^j}\left(\frac{\partial x^j}{\partial \xi^n}\frac{\partial d\xi^n}{\partial \xi^m} + d\xi^n\frac{\partial^2 x^j}{\partial \xi^m \partial \xi^n}\right)\\
&= d\xi^l\frac{\partial}{\partial \xi^l}\left(\rho d\xi^k\right) + \rho d\xi^l d\xi^n\frac{\partial \xi^k}{\partial x^i}\frac{\partial^2 x^i}{\partial \xi^l \partial \xi^n} + \rho d\xi^k\frac{\partial d\xi^m}{\partial \xi^m} + \rho d\xi^k d\xi^n\frac{\partial \xi^m}{\partial x^j}\frac{\partial^2 x^j}{\partial \xi^m \partial \xi^n}\\
&= \frac{\partial}{\partial \xi^l}\left(\rho d\xi^k d\xi^l\right) + \rho d\xi^l d\xi^n\frac{\partial \xi^k}{\partial x^i}\frac{\partial^2 x^i}{\partial \xi^l \partial \xi^n} + \rho d\xi^k d\xi^l\frac{\partial \xi^m}{\partial x^i}\frac{\partial^2 x^i}{\partial \xi^l \partial \xi^m}\\
&= J\left\{J^{-1}\frac{\partial}{\partial \xi^l}(\rho d\xi^k d\xi^l) + \rho d\xi^k d\xi^l\frac{\partial J^{-1}}{\partial \xi^l}\right\} + \rho d\xi^l d\xi^m\frac{\partial \xi^k}{\partial x^i}\frac{\partial^2 x^i}{\partial \xi^l \partial \xi^m}\\
&= J\frac{\partial}{\partial \xi^l}J^{-1}\rho d\xi^k d\xi^l + \rho d\xi^l d\xi^m \Gamma^k_{lm}, \qquad (5.23)
\end{aligned}
$$

where Γ is the Christoffel symbols of the second kind, and

$$
\begin{aligned}
\Gamma^k_{lm} &= \frac{\partial \xi^k}{\partial x^i}\frac{\partial^2 x^i}{\partial \xi^l \partial \xi^m}\\
&= \frac{1}{2}g^{kn}\left(\frac{\partial g_{mn}}{\partial \xi^l} + \frac{\partial g_{ln}}{\partial \xi^m} - \frac{\partial g_{lm}}{\partial \xi^n}\right)\\
&= \frac{1}{h_k^2}\left(h_k\frac{\partial h_k}{\partial \xi^l}\delta_{km} + h_k\frac{\partial h_k}{\partial \xi^m}\delta_{kl} - h_l\frac{\partial h_l}{\partial \xi^k}\delta_{lm}\right), \qquad (5.24)
\end{aligned}
$$

where $\{g^{kn}\}$ is inverse matrix of $\{g_{kn}\}$.

Coriolis term

$$
\frac{\partial \xi^k}{\partial x^i}\epsilon^{ijp}f^j\rho dx^p = \epsilon^{klm}\frac{1}{h_k h_l h_m}\hat{f}^l d\xi^m, \qquad (5.25)
$$

where ϵ is the Levi-Civita symbol, and

$$
\hat{f}^l = \frac{\partial \xi^l}{\partial x^j}f^j. \qquad (5.26)
$$

Pressure gradient term

$$
\begin{aligned}
\frac{\partial \xi^k}{\partial x^i}\frac{\partial p}{\partial x^i} &= \frac{\partial \xi^k}{\partial x^i}\left(\frac{\partial \xi^l}{\partial x^i}\frac{\partial p}{\partial \xi^l}\right)\\
&= \frac{1}{h_k^2}\frac{\partial x^i}{\partial \xi^k}\frac{\partial \xi^l}{\partial x^i}\frac{\partial p}{\partial \xi^l}\\
&= \frac{1}{h_k^2}\frac{\partial p}{\partial \xi^k}. \qquad (5.27)
\end{aligned}
$$

After all, momentum equation is

$$\frac{\partial}{\partial t}\rho d\xi^k + J\frac{\partial}{\partial \xi^l}\left(J^{-1}\rho d\xi^k d\xi^l\right) + \rho d\xi^l d\xi^m \Gamma^k_{lm} + \epsilon^{klm}\frac{1}{h_k h_l h_m}\hat{f}^l \rho d\xi^m$$
$$= -\frac{1}{h_k^2}\frac{\partial p}{\partial \xi^k} + \rho g^p \frac{\partial \xi^k}{\partial x^p}. \tag{5.28}$$

5.3 Map factor

We introduce Map factor m, n.

$$\frac{1}{m}\frac{a+z}{a} = h_1, \tag{5.29}$$
$$\frac{1}{n}\frac{a+z}{a} = h_2, \tag{5.30}$$
$$1 = h_3, \tag{5.31}$$

where is a is radius of the planet. Assuming shallow atmospher,

$$m = h_1^{-1}, \tag{5.32}$$
$$n = h_2^{-1}. \tag{5.33}$$

Normalized velocity is defined as

$$\hat{u} = h_1\frac{d\xi}{dt} = \frac{1}{m}\frac{d\xi}{dt}, \tag{5.34}$$
$$\hat{v} = h_2\frac{d\eta}{dt} = \frac{1}{n}\frac{d\eta}{dt}, \tag{5.35}$$
$$\hat{w} = h_3\frac{d\zeta}{dt} = \frac{d\zeta}{dt}. \tag{5.36}$$

The continuous equation becomes

$$\frac{\partial \rho}{\partial t} + mn\frac{\partial}{\partial \xi}\frac{\rho\hat{u}}{n} + mn\frac{\partial}{\partial \eta}\frac{\rho\hat{v}}{m} + \frac{\partial}{\partial \zeta}\rho\hat{w} = 0 \tag{5.37}$$

The momentum equations are

$$\frac{\partial \rho\hat{u}^k}{\partial t} + mn\frac{\partial}{\partial \xi}\frac{\rho\hat{u}\hat{u}^k}{n} + mn\frac{\partial}{\partial \eta}\frac{\rho\hat{v}\hat{u}^k}{m} + \frac{\partial}{\partial \zeta}\rho\hat{w}\hat{u}^k$$
$$+ mm_k\rho\hat{u}\hat{u}^k\frac{\partial}{\partial \xi}\frac{1}{m_k} + nm_k\rho\hat{v}\hat{u}^k\frac{\partial}{\partial \eta}\frac{1}{m_k}$$
$$- mm_k\rho\hat{u}^2\frac{\partial}{\partial \xi^k}\frac{1}{m} - nm_k\rho\hat{v}^2\frac{\partial}{\partial \xi^k}\frac{1}{n} + \epsilon^{klm}m_l\hat{f}^l\rho\hat{u}^m$$
$$= -m_k\frac{\partial p}{\partial \zeta^k} + \rho g\delta_{3k}. \tag{5.38}$$

This equation can also be written as

$$\frac{\partial \rho u}{\partial t} + mn\frac{\partial}{\partial \xi}\frac{\rho uu}{n} + mn\frac{\partial}{\partial \eta}\frac{\rho uv}{m} + \frac{\partial}{\partial \zeta}\rho uw - f\rho v - mn\rho v\left\{v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right\} = -m\frac{\partial p}{\partial \xi}, \tag{5.39}$$

$$\frac{\partial \rho v}{\partial t} + mn\frac{\partial}{\partial \xi}\frac{\rho uv}{n} + mn\frac{\partial}{\partial \eta}\frac{\rho vv}{m} + \frac{\partial}{\partial \zeta}\rho vw + f\rho u + mn\rho u\left\{v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right\} = -n\frac{\partial p}{\partial \eta}, \tag{5.40}$$

$$\frac{\partial \rho w}{\partial t} + mn\frac{\partial}{\partial \xi}\frac{\rho uw}{n} + mn\frac{\partial}{\partial \eta}\frac{\rho vw}{m} + \frac{\partial}{\partial \zeta}\rho ww = -\frac{\partial p}{\partial \zeta} - \rho g. \tag{5.41}$$

The thermodynamical and tracer equations

$$\frac{\partial \rho \phi}{\partial t} + mn\frac{\partial}{\partial \xi}\frac{\rho \hat{u}\phi}{n} + mn\frac{\partial}{\partial \eta}\frac{\rho \hat{v}\phi}{m} + \frac{\partial \rho \hat{w}\phi}{\partial \zeta} = 0. \tag{5.42}$$

Chapter 6

Horizontal explicit virtical implicit

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6.1 Equations

$$\frac{\partial G^{\frac{1}{2}}\rho}{\partial t} = -\frac{\partial J_{33}G^{\frac{1}{2}}\rho w}{\partial \xi} + G^{\frac{1}{2}}S_\rho, \tag{6.1}$$

$$\frac{\partial G^{\frac{1}{2}}\rho w}{\partial t} = -\frac{\partial J_{33}G^{\frac{1}{2}}p}{\partial \xi} - G^{\frac{1}{2}}\rho g + G^{\frac{1}{2}}S_{\rho w}, \tag{6.2}$$

$$\frac{\partial G^{\frac{1}{2}}\rho\theta}{\partial t} = -\frac{\partial J_{33}G^{\frac{1}{2}}\rho w\theta}{\partial \xi} + G^{\frac{1}{2}}S_{\rho\theta}, \tag{6.3}$$

$$p = P_{00}\left(\frac{R\rho\theta}{P_{00}}\right)^{c_p/c_v}, \tag{6.4}$$

where

$$\begin{aligned} G^{\frac{1}{2}}S_\rho &= -G^{\frac{1}{2}}\frac{\partial \rho u}{\partial x} - G^{\frac{1}{2}}\frac{\partial \rho v}{\partial y} \\ &= -\frac{\partial G^{\frac{1}{2}}\rho u}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}\rho v}{\partial y^*} - \frac{\partial J_{13}G^{\frac{1}{2}}\rho u + J_{23}G^{\frac{1}{2}}\rho v}{\partial \xi}, \end{aligned} \tag{6.5}$$

$$\begin{aligned} G^{\frac{1}{2}}S_{\rho w} &= -G^{\frac{1}{2}}\frac{\partial u\rho w}{\partial x} - G^{\frac{1}{2}}\frac{\partial v\rho w}{\partial y} - G^{\frac{1}{2}}\frac{\partial w\rho w}{\partial z} \\ &= -\frac{\partial G^{\frac{1}{2}}u\rho w}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}v\rho w}{\partial y^*} - \frac{\partial}{\partial \xi}(J_{13}G^{\frac{1}{2}}u\rho w + J_{23}G^{\frac{1}{2}}v\rho w + J_{33}G^{\frac{1}{2}}w\rho w), \end{aligned} \tag{6.6}$$

$$\begin{aligned} G^{\frac{1}{2}}S_{\rho\theta} &= -G^{\frac{1}{2}}\frac{\partial u\rho\theta}{\partial x} - G^{\frac{1}{2}}\frac{\partial v\rho\theta}{\partial y} \\ &= -\frac{\partial G^{\frac{1}{2}}u\rho\theta}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}v\rho\theta}{\partial y^*} - \frac{\partial J_{13}G^{\frac{1}{2}}u\rho\theta + J_{23}G^{\frac{1}{2}}v\rho\theta}{\partial \xi}. \end{aligned} \tag{6.7}$$

6.2 Descritization

For the temporal discritization, backward temporal integrations are employed for the terms related to acoustic wave in vertical direction.

$$\frac{\rho^{n+1}-\rho^n}{\Delta t} = -G^{-\frac{1}{2}}\frac{\partial}{\partial\xi}\{J_{33}G^{\frac{1}{2}}(\rho w)^{n+1}\} + S_\rho^n, \tag{6.8}$$

$$\frac{(\rho w)^{n+1}-(\rho w)^n}{\Delta t} = -G^{-\frac{1}{2}}\frac{\partial}{\partial\xi}(J_{33}G^{\frac{1}{2}}p^{n+1}) - g\rho^{n+1} + S_{\rho w}^n, \tag{6.9}$$

$$\frac{p^{n+1}-p^n}{\Delta t} = \frac{c_p^n}{c_v^n}\frac{p^n}{(\rho\theta)^n}\frac{\partial\rho\theta}{\partial t}, \tag{6.10}$$

$$\frac{\partial\rho\theta}{\partial t} = -G^{-\frac{1}{2}}\frac{\partial}{\partial\xi}\{J_{33}G^{\frac{1}{2}}\theta^n(\rho w)^{n+1}\} + S_{\rho\theta}^n. \tag{6.11}$$

Note that the potential temperature at previous step, θ^n, is used.

Eliminating $p^{n+1}, (\rho\theta)^{n+1}$, and ρ^{n+1}, the Helmholtz equation for $(\rho w)^{n+1}$ is obtained:

$$(\rho w)^{n+1} - \frac{\Delta t^2 g}{G^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\{J_{33}G^{\frac{1}{2}}(\rho w)^{n+1}\} - \frac{\Delta t^2}{G^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left(J_{33}\frac{c_p^n p^n}{c_v^n(\rho\theta)^n}\frac{\partial J_{33}G^{\frac{1}{2}}\theta^n(\rho w)^{n+1}}{\partial\xi}\right)$$
$$= (\rho w)^n - \frac{\Delta t}{G^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left\{J_{33}G^{\frac{1}{2}}p^n\left(1+\frac{\Delta t c_p^n S_{\rho\theta}^n}{c_v^n(\rho\theta)^n}\right)\right\} - \Delta t g(\rho^n + \Delta t S_\rho^n) + \Delta t S_{\rho w}^n. \tag{6.12}$$

Vertical differencials are discritized as follows:

$$(\rho w)_{k+1/2}^{n+1} - \frac{\Delta t^2 g}{(\Delta z_{k+1}+\Delta z_k)G_{k+1/2}^{\frac{1}{2}}}\left\{J_{33}G^{\frac{1}{2}}(\rho w)_{k+3/2}^{n+1} - J_{33}G^{\frac{1}{2}}(\rho w)_{k-1/2}^{n+1}\right\}$$
$$- \frac{\Delta t^2}{\Delta z_{k+1/2}G_{k+1/2}^{\frac{1}{2}}}\left\{\left(J_{33}\frac{c_p p}{c_v\rho\theta}\right)_{k+1}\frac{J_{33}G^{\frac{1}{2}}(\rho w)_{k+3/2}\hat{\theta}_{k+3/2} - J_{33}G^{\frac{1}{2}}(\rho w)_{k+1/2}\hat{\theta}_{k+1/2}}{\Delta z_{k+1}}\right.$$
$$\left. - \left(J_{33}\frac{c_p p}{c_v\rho\theta}\right)_k\frac{J_{33}G^{\frac{1}{2}}(\rho w)_{k+1/2}\hat{\theta}_{k+1/2} - J_{33}G^{\frac{1}{2}}(\rho w)_{k-1/2}\hat{\theta}_{k-1/2}}{\Delta z_k}\right\}$$
$$= (\rho w)_{k+1/2}^n$$
$$- \frac{\Delta t}{\Delta z_{k+1/2}G_{k+1/2}^{\frac{1}{2}}}\left\{J_{33}G^{\frac{1}{2}}p_{k+1}\left(1+\frac{\Delta t c_p S_{\rho\theta}}{c_v\rho\theta}\right)_{k+1} - J_{33}G^{\frac{1}{2}}p_k\left(1+\frac{\Delta t c_p S_{\rho\theta}}{c_v\rho\theta}\right)_k\right\}$$
$$- \frac{\Delta t g}{2}\{(\rho+\Delta t S_\rho)_{k+1} + (\rho+\Delta t S_\rho)_k\} + \Delta t S_{\rho w}, \tag{6.13}$$

where

$$\hat{\theta}_{k+1/2} = \frac{1}{12}(-\theta_{k+2} + 7\theta_{k+1} + 7\theta_k - \theta_{k-1}). \tag{6.14}$$

Finally we obtained

$$
\begin{aligned}
&-\frac{1}{G^{\frac{1}{2}}_{k+1/2}}\left\{\frac{\hat{\theta}_{k+3/2}}{\Delta z_{k+1/2}}A_{k+1}+B_{k+1/2}\right\}(\rho w)^{n+1}_{k+3/2} && (6.15)\\
&+\left\{1+\frac{\hat{\theta}_{k+1/2}}{\Delta z_{k+1/2}G^{\frac{1}{2}}_{k+1/2}}(A_{k+1}+A_k)\right\}(\rho w)^{n+1}_{k+1/2} && (6.16)\\
&-\frac{1}{G^{\frac{1}{2}}_{k+1/2}}\left\{\frac{\hat{\theta}_{k-1/2}}{\Delta z_{k+1/2}}A_{k}-B_{k+1/2}\right\}(\rho w)^{n+1}_{k-1/2} && (6.17)\\
&=C_{k+1/2}, && (6.18)
\end{aligned}
$$

where

$$
\begin{aligned}
A_k &= \frac{\Delta t^2 J_{33} G^{\frac{1}{2}}}{\Delta z_k}\left(J_{33}\frac{c_p p}{c_v \rho\theta}\right)_k, && (6.19)\\
B_{k+1/2} &= \frac{\Delta t^2 g J_{33} G^{\frac{1}{2}}}{\Delta z_{k+1}+\Delta z_k}, && (6.20)\\
C_{k+1/2} &= (\rho w)^n_{k+1/2}\\
&-\Delta t\frac{J_{33}G^{\frac{1}{2}}p_{k+1}\left(1+\Delta t\frac{c_p S_{\rho\theta}}{c_v\rho\theta}\right)_{k+1}-J_{33}G^{\frac{1}{2}}p_k\left(1+\Delta t\frac{c_p S_{\rho\theta}}{c_v\rho\theta}\right)_k}{\Delta z_{k+1/2}G^{\frac{1}{2}}_{k+1/2}}\\
&-\Delta t g\frac{(\rho+\Delta t S_\rho)_{k+1}+(\rho+\Delta t S_\rho)_k}{2}+\Delta t S_{\rho w}. && (6.21)
\end{aligned}
$$

Chapter 7

Horizontally and virtically implicit

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7.1 Equations

The governing equation is the followings:

$$\frac{\partial G^{\frac{1}{2}}\rho'}{\partial t} = -\frac{\partial G^{\frac{1}{2}}\rho u}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}\rho v}{\partial y^*} - \frac{\partial J_{33}G^{\frac{1}{2}}\rho w}{\partial \xi} + G^{\frac{1}{2}}S_\rho, \tag{7.1}$$

$$\frac{\partial G^{\frac{1}{2}}\rho u}{\partial t} = -\frac{\partial G^{\frac{1}{2}}p'}{\partial x^*} + G^{\frac{1}{2}}S_{\rho u}, \tag{7.2}$$

$$\frac{\partial G^{\frac{1}{2}}\rho v}{\partial t} = -\frac{\partial G^{\frac{1}{2}}p'}{\partial y^*} + G^{\frac{1}{2}}S_{\rho v}, \tag{7.3}$$

$$\frac{\partial G^{\frac{1}{2}}\rho w}{\partial t} = -\frac{\partial J_{33}G^{\frac{1}{2}}p'}{\partial \xi} - G^{\frac{1}{2}}\rho' g + G^{\frac{1}{2}}S_{\rho w}, \tag{7.4}$$

$$\frac{\partial G^{\frac{1}{2}}\rho\theta}{\partial t} = -\frac{\partial G^{\frac{1}{2}}u\rho\theta}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}v\rho\theta}{\partial y^*} - \frac{\partial J_{33}G^{\frac{1}{2}}w\rho\theta}{\partial \xi} + G^{\frac{1}{2}}S_{\rho\theta}, \tag{7.5}$$

$$p = P_{00}\left(\frac{R\rho\theta}{P_{00}}\right)^{c_p/c_v}, \tag{7.6}$$

where

$$G^{\frac{1}{2}}S_\rho = -\frac{\partial J_{13}G^{\frac{1}{2}}\rho u + J_{23}G^{\frac{1}{2}}\rho v}{\partial \xi}, \tag{7.7}$$

$$G^{\frac{1}{2}}S_{\rho u} = -\frac{\partial G^{\frac{1}{2}}u\rho u}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}v\rho u}{\partial y^*} - \frac{\partial}{\partial \xi}(J_{13}G^{\frac{1}{2}}u\rho u + J_{23}G^{\frac{1}{2}}v\rho u + J_{33}G^{\frac{1}{2}}w\rho u)$$
$$- \frac{\partial}{\partial \xi}(J_{13}G^{\frac{1}{2}}p'), \tag{7.8}$$

$$G^{\frac{1}{2}}S_{\rho v} = -\frac{\partial G^{\frac{1}{2}}u\rho v}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}v\rho v}{\partial y^*} - \frac{\partial}{\partial \xi}(J_{13}G^{\frac{1}{2}}u\rho v + J_{23}G^{\frac{1}{2}}v\rho v + J_{33}G^{\frac{1}{2}}w\rho v),$$
$$- \frac{\partial}{\partial \xi}(J_{23}G^{\frac{1}{2}}p'), \tag{7.9}$$

$$G^{\frac{1}{2}}S_{\rho w} = -\frac{\partial G^{\frac{1}{2}}u\rho w}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}v\rho w}{\partial y^*} - \frac{\partial}{\partial \xi}(J_{13}G^{\frac{1}{2}}u\rho w + J_{23}G^{\frac{1}{2}}v\rho w + J_{33}G^{\frac{1}{2}}w\rho w), \tag{7.10}$$

$$G^{\frac{1}{2}}S_{\rho\theta} = -\frac{\partial J_{13}G^{\frac{1}{2}}u\rho\theta + J_{23}G^{\frac{1}{2}}v\rho\theta}{\partial \xi}. \tag{7.11}$$

Prime describes deviation from a reference state, and the reference state depends only z and satisfies in hydrostatic barance:

$$p' = p - \bar{p}, \tag{7.12}$$
$$\rho' = \rho - \bar{\rho}, \tag{7.13}$$
$$\frac{d\bar{p}(z)}{dz} = -\bar{\rho}(z)g. \tag{7.14}$$

7.2 Descritization

For the temporal discritization, backward temporal integrations are employed for the terms related to acoustic wave.

$$\frac{\rho'^{n+1} - \rho'^{n}}{\Delta t} = -G^{-\frac{1}{2}}\frac{\partial}{\partial x^*}\{G^{\frac{1}{2}}(\rho u)^{n+1}\} - G^{-\frac{1}{2}}\frac{\partial}{\partial y^*}\{G^{\frac{1}{2}}(\rho v)^{n+1}\}$$
$$- G^{-\frac{1}{2}}\frac{\partial}{\partial \xi}\{J_{33}G^{\frac{1}{2}}(\rho w)^{n+1}\} + S_\rho^n, \tag{7.15}$$

$$\frac{(\rho u)^{n+1} - (\rho u)^n}{\Delta t} = -G^{-\frac{1}{2}}\frac{\partial}{\partial x^*}(G^{\frac{1}{2}}p'^{n+1}) + S_{\rho u}^n, \tag{7.16}$$

$$\frac{(\rho v)^{n+1} - (\rho v)^n}{\Delta t} = -G^{-\frac{1}{2}}\frac{\partial}{\partial y^*}(G^{\frac{1}{2}}p'^{n+1}) + S_{\rho v}^n, \tag{7.17}$$

$$\frac{(\rho w)^{n+1} - (\rho w)^n}{\Delta t} = -G^{-\frac{1}{2}}\frac{\partial}{\partial \xi}(J_{33}G^{\frac{1}{2}}p'^{n+1}) - g\rho'^{n+1} + S_{\rho w}^n, \tag{7.18}$$

$$\frac{p'^{n+1} - p'^{n}}{\Delta t} = P_{00}\left(\frac{R}{P_{00}}\right)^{\kappa^n}\kappa^n(\rho\theta)^{n\kappa^n - 1}\frac{\partial \rho\theta}{\partial t} = \kappa^n\frac{p^n}{(\rho\theta)^n}\frac{\partial \rho\theta}{\partial t}, \tag{7.19}$$

$$\frac{\partial \rho\theta}{\partial t} = -G^{-\frac{1}{2}}\frac{\partial}{\partial x^*}\{G^{\frac{1}{2}}\theta^n(\rho u)^{n+1}\} - G^{-\frac{1}{2}}\frac{\partial}{\partial y^*}\{G^{\frac{1}{2}}\theta^n(\rho v)^{n+1}\}$$
$$- G^{-\frac{1}{2}}\frac{\partial}{\partial \xi}\{J_{33}G^{\frac{1}{2}}\theta^n(\rho w)^{n+1}\} + S_{\rho\theta}^n, \tag{7.20}$$

where $\kappa = c_p/c_v$. Note that the potential temperature at previous step, θ^n, is used.

In order to obtain Helmholtz equation, a linearlized equation for the density is used instead of Eq. 7.15.

$$\rho'^{n+1} \sim \rho'^{n} + \frac{1}{\kappa^n}\frac{\rho^n}{p^n}\left\{p'^{n+1} - p'^{n}\right\}. \tag{7.21}$$

Here we assume that the potential temperature does not change during a temporal step due to acoustic wave.

Eliminating $(\rho u)^{n+1}, (\rho v)^{n+1} (\rho w)^{n+1},$, and ρ'^{n+1}, the Helmholtz equation for p'^{n+1} is obtained.

$$\begin{aligned}
&\frac{\partial}{\partial x^*}\left(\theta^n \frac{\partial G^{\frac{1}{2}} p'^{n+1}}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(\theta^n \frac{\partial G^{\frac{1}{2}} p'^{n+1}}{\partial y^*}\right) + \frac{\partial}{\partial \xi}\left(J_{33}\theta^n \frac{\partial J_{33} G^{\frac{1}{2}} p'^{n+1}}{\partial \xi}\right) \\
&+ g\frac{\partial}{\partial \xi}\left(\frac{J_{33} G^{\frac{1}{2}} \theta^n p'^{n+1}}{C_s^{2n}}\right) - \frac{G^{\frac{1}{2}}\theta^n p'^{n+1}}{\Delta t^2 C_s^{2n}} \\
&= \frac{1}{\Delta t}\left[\frac{\partial G^{\frac{1}{2}}\theta^n\left\{(\rho u)^n + \Delta t S_{\rho u}^n\right\}}{\partial x^*} + \frac{\partial G^{\frac{1}{2}}\theta^n\left\{(\rho v)^n + \Delta t S_{\rho v}^n\right\}}{\partial y^*} + \frac{\partial J_{33} G^{\frac{1}{2}}\theta^n\left\{(\rho w)^n + \Delta t S_{\rho w}^n\right\}}{\partial \xi}\right] \\
&\quad + g\frac{\partial}{\partial \xi}\left\{\frac{J_{33} G^{\frac{1}{2}}\theta^n p'^n}{C_s^{2n}} - J_{33} G^{\frac{1}{2}}(\rho'\theta)^n\right\} + \frac{G^{\frac{1}{2}} S_{\rho\theta}}{\Delta t} - \frac{G^{\frac{1}{2}}\theta^n p'^n}{\Delta t^2 C_s^{2n}},
\end{aligned} \tag{7.22}$$

where

$$C_s^2 = \kappa\frac{p}{\rho}. \tag{7.23}$$

Spatial differentials are descritized.

$$
\begin{aligned}
&\frac{1}{\Delta x_i}\left(\hat{\theta}_{i+1/2}\frac{(G^{\frac{1}{2}}p'^{n+1})_{i+1}-G^{\frac{1}{2}}p'^{n+1}}{\Delta x_{i+1/2}}-\hat{\theta}_{i-1/2}\frac{G^{\frac{1}{2}}p'^{n+1}-(G^{\frac{1}{2}}p'^{n+1})_{i-1}}{\Delta x_{i-1/2}}\right)\\
&+\frac{1}{\Delta y_j}\left(\hat{\theta}_{j+1/2}\frac{(G^{\frac{1}{2}}p'^{n+1})_{j+1}-G^{\frac{1}{2}}p'^{n+1}}{\partial y_{j+1/2}}-\hat{\theta}_{j-1/2}\frac{G^{\frac{1}{2}}p'^{n+1}-(G^{\frac{1}{2}}p'^{n+1})_{j-1}}{\partial y_{j-1/2}}\right)\\
&+\frac{1}{\Delta z_k}\left\{(J_{33})_{k+1/2}\hat{\theta}_{k+1/2}\frac{J_{33}G^{\frac{1}{2}}p'^{n+1}_{k+1}-J_{33}G^{\frac{1}{2}}p'^{n+1}}{\Delta z_{k+1/2}}-(J_{33})_{k-1/2}\hat{\theta}_{k-1/2}\frac{J_{33}G^{\frac{1}{2}}p'^{n+1}-J_{33}G^{\frac{1}{2}}p'^{n+1}_{k-1}}{\Delta z_{k-1/2}}\right)\\
&+g\frac{1}{\Delta_{k+1/2}+\Delta z_{k-1/2}}\left\{\frac{J_{33}G^{\frac{1}{2}}\theta p'^{n+1}_{k+1}}{C^2_{sk+1}}-\frac{J_{33}G^{\frac{1}{2}}\theta p'^{n+1}_{k-1}}{C^2_{sk-1}}\right\}-\frac{G^{\frac{1}{2}}\theta p'^{n+1}}{\Delta t^2 C_s^2}\\
&=\frac{1}{\Delta t}\left[\frac{G^{\frac{1}{2}}_{i+1/2}\hat{\theta}_{i+1/2}\left\{(\rho u)_{i+1/2}+\Delta t(S_{\rho u})_{i+1/2}\right\}-G^{\frac{1}{2}}_{i-1/2}\hat{\theta}_{i-1/2}\left\{(\rho u)_{i-1/2}+\Delta t(S_{\rho u})_{i-1/2}\right\}}{\Delta x_i}\right.\\
&+\frac{G^{\frac{1}{2}}_{j+1/2}\hat{\theta}_{j+1/2}\left\{(\rho v)_{j+1/2}+\Delta t(S_{\rho v})_{j+1/2}\right\}-G^{\frac{1}{2}}_{j-1/2}\hat{\theta}_{j-1/2}\left\{(\rho v)_{j-1/2}+\Delta t' S_{\rho v})_{j-1/2}\right\}}{\Delta y_j}\\
&\left.+\frac{J_{33}G^{\frac{1}{2}}\hat{\theta}_{k+1/2}\left\{(\rho w)_{k+1/2}+\Delta t(S_{\rho w})_{k+1/2}\right\}-J_{33}G^{\frac{1}{2}}\hat{\theta}_{k-1/2}\left\{(\rho w)_{k-1/2}+\Delta t(S_{\rho w})_{k-1/2}\right\}}{\Delta z_k}\right]\\
&\quad+g\frac{1}{\Delta z_{k+1/2}+\Delta z_{k-1/2}}\left\{\frac{J_{33}G^{\frac{1}{2}}\theta p'^{n}_{k+1}}{C^2_{sk+1}}-\frac{J_{33}G^{\frac{1}{2}}\theta p'^{n}_{k-1}}{C^2_{sk-1}}-J_{33}G^{\frac{1}{2}}\{(\rho'\theta)_{k+1}-(\rho'\theta)_{k-1}\}\right\}\\
&+\frac{G^{\frac{1}{2}}S_{\rho\theta}}{\Delta t}-\frac{G^{\frac{1}{2}}\theta p'^{n}}{\Delta t^2 C_s^2}. \qquad (7.24)
\end{aligned}
$$

Chapter 8

Physical parameterization

8.1 Turbulence

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8.1.1 Spatial filter

The governing equations are the following:

$$\frac{\partial \rho}{\partial t} + \frac{\partial u_i \rho}{\partial x_i} = 0 \tag{8.1}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial u_j \rho u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + g\rho\delta_{i3} \tag{8.2}$$

$$\frac{\partial \rho \theta}{\partial t} + \frac{\partial u_i \rho \theta}{\partial x_i} = Q \tag{8.3}$$

Spatially filtering the continuity equation yields:

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{u_i \rho}}{\partial x_i} = 0, \tag{8.4}$$

where $\overline{\phi}$ indicates the spatially filtered quantity of an arbitrary variable ϕ. Favre filtering (Favre, 1983), defined by:

$$\widetilde{\phi} = \frac{\overline{\rho \phi}}{\overline{\rho}} \tag{8.5}$$

renders the equation (8.4):

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \widetilde{u_i} \overline{\rho}}{\partial x_i} = 0. \tag{8.6}$$

The momentum equations become:

$$\frac{\partial \overline{\rho u_i}}{\partial t} + \frac{\partial \overline{u_j \rho u_i}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \overline{\rho} g \delta_{i3} \tag{8.7}$$

$$\frac{\partial \overline{\rho}\widetilde{u_i}}{\partial t} + \frac{\partial \widetilde{u_j}\,\overline{\rho}\widetilde{u_i}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + g\overline{\rho}\delta_{i3} - \frac{\partial}{\partial x_j}\left(\overline{u_i \rho u_j} - \widetilde{u_j}\overline{\rho}\widetilde{u_i}\right) \tag{8.8}$$

$$\frac{\partial \overline{\rho}\widetilde{u_i}}{\partial t} + \frac{\partial \widetilde{u_j}\,\overline{\rho}\widetilde{u_i}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + g\overline{\rho}\delta_{i3} - \frac{\partial}{\partial x_j}\overline{\rho}\left(\widetilde{u_i u_j} - \widetilde{u_j}\widetilde{u_i}\right). \tag{8.9}$$

As the same matter, the thermal equation becomes:

$$\frac{\partial \overline{\rho}\widetilde{\theta}}{\partial t} + \frac{\partial \widetilde{u}_i \overline{\rho}\widetilde{\theta}}{\partial x_i} = Q - \frac{\partial}{\partial x_i}\overline{\rho}\left(\widetilde{u_i \theta} - \widetilde{u}_i \widetilde{\theta}\right). \tag{8.10}$$

The govering equations for the prognostic variables ($\overline{\rho}, \overline{\rho}\widetilde{u}_i$, and $\overline{\rho}\widetilde{\theta}$) are:

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \widetilde{u}_i \overline{\rho}}{\partial x_i} = 0, \tag{8.11}$$

$$\frac{\partial \overline{\rho}\widetilde{u}_i}{\partial t} + \frac{\partial \widetilde{u}_j \overline{\rho}\widetilde{u}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + g\overline{\rho}\delta_{i3} - \frac{\partial \overline{\rho}\tau_{ij}}{\partial x_j}, \tag{8.12}$$

$$\frac{\partial \overline{\rho}\widetilde{\theta}}{\partial t} + \frac{\partial \widetilde{u}_i \overline{\rho}\widetilde{\theta}}{\partial x_i} = Q - \frac{\partial \overline{\rho}\tau_i^D}{\partial x_i}, \tag{8.13}$$

where:

$$\tau_{ij} = \widetilde{u_i u_j} - \widetilde{u}_i \widetilde{u}_j, \tag{8.14}$$

$$\tau_i^D = \widetilde{u_i \theta} - \widetilde{u}_i \widetilde{\theta}. \tag{8.15}$$

Hereafter, we omit the overline and tilde representing the spatial and Favre filters.

8.1.2 SGS model

Smagorinsky-Lilly model

The eddy momentum flux is:

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = -2\nu_{SGS}\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right), \tag{8.16}$$

where S_{ij} is the strain tensor:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{8.17}$$

and:

$$\nu_{SGS} = (C_s \lambda)^2 |S|. \tag{8.18}$$

C_s is the Smagorinsky constant, λ is a characteristic SGS length scale, and $|S|$ is scale of the tensor S,

$$|S| = \sqrt{2S_{ij}S_{ij}}. \tag{8.19}$$

The eddy momentum flux is then:

$$\tau_{ij} = -2\nu_{SGS}\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right) + \frac{2}{3}TKE\delta_{ij}, \tag{8.20}$$

where:

$$TKE = \frac{1}{2}\tau_{ii} = \left(\frac{\nu_{SGS}}{C_k \lambda}\right)^2, \tag{8.21}$$

where C_k is a SGS constant and assumed to be 0.1, following Deardorff (1980) and Moeng and Wyngaard (1988).

The eddy heat flux is:

$$\tau_i^D = -D_{SGS} \frac{\partial \theta}{\partial x_i}, \tag{8.22}$$

where:

$$D_{SGS} = \frac{1}{Pr} \nu_{SGS}. \tag{8.23}$$

Pr is the turbulent Prandtl number. For other scalar constants such as water vapor, D_{SGS} is also used as their diffusivity.

To include buoyancy effects, the extension of the basic Smagorinsky constant developed by Brown et al. (1994) is used.

$$\nu_{SGS} = (C_s \lambda)^2 |S| \sqrt{1 - Rf}, \tag{8.24}$$

where Rf is the flux Richardson number ($Rf = Ri/Pr$). Ri is the local (pointwise) gradient Richardson number,

$$Ri = \frac{N^2}{|S|^2}, \tag{8.25}$$

and N^2 is the Brunt-Visala frequency,

$$N^2 = \frac{g}{\theta} \frac{\partial \theta}{\partial z}. \tag{8.26}$$

The Prandtl number is an unknown parameter that depends on the Richardson number, though it is often assumed to have a constant value. For unstable conditions ($Ri < 0$),

$$\nu_{SGS} = (C_s \lambda)^2 |S| \sqrt{1 - cRi}, \tag{8.27}$$

$$D_{SGS} = \frac{1}{Pr_N} (C_s \lambda)^2 |S| \sqrt{1 - bRi}, \tag{8.28}$$

where Pr_N is the Prandtl number for neutral condtions. The values of c, b, Pr_N are set to 16, 40, and 0.7, respectively. The Prandtl number is then:

$$Pr = Pr_N \sqrt{\frac{1 - cRi}{1 - bRi}}. \tag{8.29}$$

For stable conditions, when the Richardson number is smaller than the critical Richardson number, $Ri_c (= 0.25)$,

$$\nu_{SGS} = (C_s \lambda)^2 |S| \left(1 - \frac{Ri}{Ri_c}\right)^4, \tag{8.30}$$

$$D_{SGS} = \frac{1}{Pr_N} (C_s \lambda)^2 |S| \left(1 - \frac{Ri}{Ri_c}\right)^4 (1 - gRi). \tag{8.31}$$

The constant g is determined as the Prandtl number becomes 1 in the limit of $Ri \to Ri_C$ and is then $(1 - Pr_N)/Ri_c$. The Prandtl number is

$$Pr = Pr_N \left\{1 - (1 - Pr_N) \frac{Ri}{Ri_c}\right\}^{-1}. \tag{8.32}$$

For strongly stable conditions ($Ri > Ri_c$), eddy viscosity and diffusivity for scalars are 0;

$$\nu_{SGS} = 0, \tag{8.33}$$

$$D_{SGS} = 0. \tag{8.34}$$

The Prandtl number is $Pr = 1$.

Scotti et al. (1993) suggested that the length scale should depend on the grid aspect ratio. Under equilibrium conditions with the universal Kolmogorov spectrum, energy cascaded to the SGS turbulence, which is equal to SGS dissipation, must not depend on the grid aspect ratio. The energy flux or dissipation can be written as function of S_{ij} and the length scale, λ. The S_{ij} depends on the grid aspect ratio, so the length scale should have dependency on the aspect ratio, cancelling the dependency of S_{ij}. With some approximations, the authors obtained an approximate function of the length scale [1] :

$$\lambda = f(a)\Delta, \tag{8.35}$$

where $f(a)$ is a function of grid aspect ratio, a, and

$$\begin{aligned} f(a) = 1.736a^{1/3}\{ & \\ & 4P_1(b_1)a^{1/3} + 0.222P_2(b_1)a^{-5/3} + 0.077P_3(b_1)a^{-11/3} \\ & - 3b_1 + 4P_1(b_2) + 0.222P_2(b_2) + 0.077P_3(b_2) - 3b_2 \\ & \}^{-3/4}. \end{aligned} \tag{8.36}$$

Here $b_1 = \arctan(1/a), b_2 = \arctan(a) = \pi/2 - b_1$, and

$$P_1(z) = 2.5P_2(z) - 1.5(\cos(z))^{2/3}\sin(z), \tag{8.37}$$

$$P_2(z) = 0.98z + 0.073z^2 - 0.418z^3 + 0.120z^4, \tag{8.38}$$

$$P_3(z) = 0.976z + 0.188z^2 + 1.169z^3 + 0.755z^4 - 0.151z^5. \tag{8.39}$$

For instance, $f(2) = 1.036, f(5) = 1.231, f(10) = 1.469$, and $f(20) = 1.790$. Δ is the filter length, and is here defined to be proportional to $(\Delta x\Delta y\Delta z)^{1/3}$. In this model, we introduce a numerical filter to reduce two-grid noise discussed above. This filter also reduces two-grid scale physical variability. This means that two-grid scale would be preferred for the filter length in this model rather than grid spacing itself; that is:

$$\Delta = 2(\Delta x\Delta y\Delta z)^{1/3}. \tag{8.40}$$

[1] They considered two grid aspect ratios, while we consider only one, i.e. $\Delta x = \Delta y$.

Terrain-following coordinates

Tendencies representing effect of sub-grid scale turbulence with terrain-following coordinates are as follows: [2];

$$\frac{\partial G^{\frac{1}{2}}\rho u}{\partial t} = -\frac{\partial G^{\frac{1}{2}}\rho\tau_{11}}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}\rho\tau_{12}}{\partial y^*} - \frac{\partial J_{13}G^{\frac{1}{2}}\rho\tau_{11} + J_{23}G^{\frac{1}{2}}\rho\tau_{12} + J_{33}G^{\frac{1}{2}}\rho\tau_{13}}{\partial\xi}, \tag{8.41}$$

$$\frac{\partial G^{\frac{1}{2}}\rho v}{\partial t} = -\frac{\partial G^{\frac{1}{2}}\rho\tau_{21}}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}\rho\tau_{22}}{\partial y^*} - \frac{\partial J_{13}G^{\frac{1}{2}}\rho\tau_{21} + J_{23}G^{\frac{1}{2}}\rho\tau_{22} + J_{33}G^{\frac{1}{2}}\rho\tau_{23}}{\partial\xi}, \tag{8.42}$$

$$\frac{\partial G^{\frac{1}{2}}\rho w}{\partial t} = -\frac{\partial G^{\frac{1}{2}}\rho\tau_{31}}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}\rho\tau_{32}}{\partial y^*} - \frac{\partial J_{13}G^{\frac{1}{2}}\rho\tau_{31} + J_{23}G^{\frac{1}{2}}\rho\tau_{32} + J_{33}G^{\frac{1}{2}}\rho\tau_{33}}{\partial\xi}, \tag{8.43}$$

$$\frac{\partial G^{\frac{1}{2}}\rho\theta}{\partial t} = -\frac{\partial G^{\frac{1}{2}}\rho\tau_1^D}{\partial x^*} - \frac{\partial G^{\frac{1}{2}}\rho\tau_2^D}{\partial y^*} - \frac{\partial J_{13}G^{\frac{1}{2}}\rho\tau_1^D + J_{23}G^{\frac{1}{2}}\rho\tau_2^D + J_{33}G^{\frac{1}{2}}\rho\tau_3^D}{\partial\xi} \tag{8.44}$$

$$G^{\frac{1}{2}}S_{11} = \frac{\partial G^{\frac{1}{2}}u}{\partial x^*} + \frac{\partial J_{13}G^{\frac{1}{2}}u}{\partial\xi}, \tag{8.45}$$

$$G^{\frac{1}{2}}S_{22} = \frac{\partial G^{\frac{1}{2}}v}{\partial y^*} + \frac{\partial J_{23}G^{\frac{1}{2}}v}{\partial\xi}, \tag{8.46}$$

$$G^{\frac{1}{2}}S_{33} = \frac{\partial J_{33}G^{\frac{1}{2}}w}{\partial\xi}, \tag{8.47}$$

$$G^{\frac{1}{2}}S_{12} = \frac{1}{2}\left(\frac{\partial G^{\frac{1}{2}}u}{\partial y^*} + \frac{\partial G^{\frac{1}{2}}v}{\partial x^*} + \frac{\partial J_{23}G^{\frac{1}{2}}u + J_{13}G^{\frac{1}{2}}v}{\partial\xi}\right), \tag{8.48}$$

$$G^{\frac{1}{2}}S_{23} = \frac{1}{2}\left(\frac{\partial G^{\frac{1}{2}}w}{\partial y^*} + \frac{\partial J_{33}G^{\frac{1}{2}}v + J_{23}G^{\frac{1}{2}}w}{\partial\xi}\right), \tag{8.49}$$

$$G^{\frac{1}{2}}S_{31} = \frac{1}{2}\left(\frac{\partial G^{\frac{1}{2}}w}{\partial x^*} + \frac{\partial J_{13}G^{\frac{1}{2}}w + J_{33}G^{\frac{1}{2}}u}{\partial\xi}\right), \tag{8.50}$$

$$G^{\frac{1}{2}}\tau_1^D = -D_{SGS}\left(\frac{\partial G^{\frac{1}{2}}\theta}{\partial x^*} + \frac{\partial J_{13}G^{\frac{1}{2}}\theta}{\partial\xi}\right), \tag{8.51}$$

$$G^{\frac{1}{2}}\tau_2^D = -D_{SGS}\left(\frac{\partial G^{\frac{1}{2}}\theta}{\partial y^*} + \frac{\partial J_{23}G^{\frac{1}{2}}\theta}{\partial\xi}\right), \tag{8.52}$$

$$G^{\frac{1}{2}}\tau_3^D = -D_{SGS}\frac{\partial J_{33}G^{\frac{1}{2}}\theta}{\partial\xi}, \tag{8.53}$$

$$G^{\frac{1}{2}}N^2 = \frac{g}{\theta}\frac{\partial J_{33}G^{\frac{1}{2}}\theta}{\partial\xi}. \tag{8.54}$$

[2]Equationsthat are not changed in the terrain-following coordinates are ommited.

8.1.3 Discretization

Spatial discretization

We use the 4th order difference scheme for the advection term, as mentioned in the chapter 3. The τ_{ij} and τ_i^D are proportional to the square of the grid spacing (Δ^2). Due to consistency with the advection term in terms of order for spatial difference, the second order central difference scheme is used for terms of sub-grid scale turbulence. In the following part of this sub-section, overline, and i, j, k have the same meaning as in the chapter 3.

Momentum equation The tendencies in the momentum equation related to the sub-grid scale mode are:

$$\begin{aligned}
\left.\frac{\partial G^{\frac{1}{2}}\rho u}{\partial t}\right|_{i+\frac{1}{2},j,k} =& - \frac{(G^{\frac{1}{2}}\rho\tau_{11})_{i+1,j,k} - (G^{\frac{1}{2}}\rho\tau_{11})_{i,j,k}}{\Delta x} \\
& - \frac{(G^{\frac{1}{2}}\overline{\rho}\tau_{12})_{i+\frac{1}{2},j+\frac{1}{2},k} - (G^{\frac{1}{2}}\overline{\rho}\tau_{12})_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta y} \\
& - \frac{\{G^{\frac{1}{2}}\overline{\rho}(J_{13}\tau_{11} + J_{23}\tau_{12} + J_{33}\tau_{13})\}_{i+\frac{1}{2},j,k+\frac{1}{2}} - \{G^{\frac{1}{2}}\overline{\rho}(J_{13}\tau_{11} + J_{23}\tau_{12} + J_{33}\tau_{13})\}_{i+\frac{1}{2},j,k-\frac{1}{2}}}{\Delta z},
\end{aligned} \tag{8.55}$$

$$\begin{aligned}
\left.\frac{\partial G^{\frac{1}{2}}\rho v}{\partial t}\right|_{i,j+\frac{1}{2},k} =& - \frac{(G^{\frac{1}{2}}\overline{\rho}\tau_{21})_{i+\frac{1}{2},j+\frac{1}{2},k} - (G^{\frac{1}{2}}\overline{\rho}\tau_{21})_{i-\frac{1}{2},j+\frac{1}{2},k}}{\Delta x} \\
& - \frac{(G^{\frac{1}{2}}\rho\tau_{22})_{i,j+1,k} - (G^{\frac{1}{2}}\rho\tau_{22})_{i,j,k}}{\Delta y} \\
& - \frac{\{G^{\frac{1}{2}}\overline{\rho}(J_{13}\tau_{21} + J_{23}\tau_{22} + J_{33}\tau_{23})\}_{i,j+\frac{1}{2},k+\frac{1}{2}} - \{G^{\frac{1}{2}}\overline{\rho}(J_{13}\tau_{21} + J_{23}\tau_{22} + J_{33}\tau_{23})\}_{i,j+\frac{1}{2},k-\frac{1}{2}}}{\Delta z},
\end{aligned} \tag{8.56}$$

$$\begin{aligned}
\left.\frac{\partial G^{\frac{1}{2}}\rho w}{\partial t}\right|_{i,j,k+\frac{1}{2}} =& - \frac{(G^{\frac{1}{2}}\overline{\rho}\tau_{31})_{i+\frac{1}{2},j,k+\frac{1}{2}} - (G^{\frac{1}{2}}\overline{\rho}\tau_{31})_{i-\frac{1}{2},j,k+\frac{1}{2}}}{\Delta x} \\
& - \frac{(G^{\frac{1}{2}}\overline{\rho}\tau_{32})_{i,j+\frac{1}{2},k+\frac{1}{2}} - (G^{\frac{1}{2}}\overline{\rho}\tau_{32})_{i,j-\frac{1}{2},k+\frac{1}{2}}}{\Delta y} \\
& - \frac{\{(G^{\frac{1}{2}}\rho(J_{13}\tau_{31} + J_{23}\tau_{32} + J_{33}\tau_{33})\}_{i,j,k+1} - \{(G^{\frac{1}{2}}\rho(J_{13}\tau_{31} + J_{23}\tau_{32} + J_{33}\tau_{33})\}_{i,j,k}}{\Delta z}.
\end{aligned} \tag{8.57}$$

The $\overline{\rho}$ is:

$$\overline{\rho}_{i,j+\frac{1}{2},k+\frac{1}{2}} = \frac{\rho_{i,j+1,k+1} + \rho_{i,j+1,k} + \rho_{i,j,k+1} + \rho_{i,j,k}}{4}, \tag{8.58}$$

$$\overline{\rho}_{i+\frac{1}{2},j,k+\frac{1}{2}} = \frac{\rho_{i+1,j,k+1} + \rho_{i+1,j,k} + \rho_{i,j,k+1} + \rho_{i,j,k}}{4}, \tag{8.59}$$

$$\overline{\rho}_{i+\frac{1}{2},j+\frac{1}{2},k} = \frac{\rho_{i+1,j+1,k} + \rho_{i+1,j,k} + \rho_{i,j+1,k} + \rho_{i,j,k}}{4}. \tag{8.60}$$

Thermal equation The tendency in the thermal equation related to the sub-grid scale model is:

$$\left.\frac{\partial G^{\frac{1}{2}}\rho\theta}{\partial t}\right|_{i,j,k} = -\frac{(G^{\frac{1}{2}}\overline{\rho}\tau_1^D)_{i+\frac{1}{2},j,k} - (G^{\frac{1}{2}}\overline{\rho}\tau_1^D)_{i-\frac{1}{2},j,k}}{\Delta x} - \frac{(G^{\frac{1}{2}}\overline{\rho}\tau_2^D)_{i,j+\frac{1}{2},k} - (G^{\frac{1}{2}}\overline{\rho}\tau_2^D)_{i,j-\frac{1}{2},k}}{\Delta y} - \frac{\{G^{\frac{1}{2}}\overline{\rho}(J_{13}\tau_1^D + J_{23}\tau_2^D + J_{33}\tau_3^D)\}_{i,j,k+\frac{1}{2}} - \{G^{\frac{1}{2}}\overline{\rho}(J_{13}\tau_1^D + J_{23}\tau_2^D + J_{33}\tau_3^D)\}_{i,j,k-\frac{1}{2}}}{\Delta z}. \tag{8.61}$$

The $\overline{\rho}$ at half-level is eq.(3.81)-(3.83).

The eddy diffusion flux, τ^D, at half-level is:

$$(G^{\frac{1}{2}}\tau_1^D)_{i+\frac{1}{2},j,k} = -D_{SGS,i+\frac{1}{2},j,k}\left\{\frac{(G^{\frac{1}{2}}\theta)_{i+1,j,k} - (G^{\frac{1}{2}}\theta)_{i,j,k}}{\Delta x} + \frac{(J_{13}G^{\frac{1}{2}}\overline{\theta})_{i+\frac{1}{2},j,k+\frac{1}{2}} - (J_{13}G^{\frac{1}{2}}\overline{\theta})_{i+\frac{1}{2},j,k-\frac{1}{2}}}{\Delta z}\right\}, \tag{8.62}$$

$$(G^{\frac{1}{2}}\tau_2^D)_{i,j+\frac{1}{2},k} = -D_{SGS,i,j+\frac{1}{2},k}\left\{\frac{(G^{\frac{1}{2}}\theta)_{i,j+1,k} - (G^{\frac{1}{2}}\theta)_{i,j,k}}{\Delta y} + \frac{(J_{23}G^{\frac{1}{2}}\overline{\theta})_{i,j+\frac{1}{2},k+\frac{1}{2}} - (J_{23}G^{\frac{1}{2}}\overline{\theta})_{i,j+\frac{1}{2},k-\frac{1}{2}}}{\Delta z}\right\}, \tag{8.63}$$

$$(G^{\frac{1}{2}}\tau_3^D)_{i,j,k+\frac{1}{2}} = -D_{SGS,i,j,k+\frac{1}{2}}\frac{J_{33}G^{\frac{1}{2}}\theta_{i,j,k+1} - J_{33}G^{\frac{1}{2}}\theta_{i,j,k}}{\Delta z}. \tag{8.64}$$

Strain tensor All the strain tensors, eq.(8.17), have to be calculated at full-level (grid cell center), and some are at cell edges.

- cell center (i,j,k)

$$(G^{\frac{1}{2}}S_{11})_{i,j,k} = \frac{(G^{\frac{1}{2}}\overline{u})_{i+\frac{1}{2},j,k} - (G^{\frac{1}{2}}\overline{u})_{i-\frac{1}{2},j,k}}{\Delta x} + \frac{(J_{13}G^{\frac{1}{2}}\overline{u})_{i+\frac{1}{2},j,k+\frac{1}{2}} - (J_{13}G^{\frac{1}{2}}\overline{u})_{i+\frac{1}{2},j,k-\frac{1}{2}}}{\Delta z}, \tag{8.65}$$

$$(G^{\frac{1}{2}}S_{22})_{i,j,k} = \frac{(G^{\frac{1}{2}}\overline{v})_{i,j+\frac{1}{2},k} - (G^{\frac{1}{2}}\overline{v})_{i,j-\frac{1}{2},k}}{\Delta y} + \frac{(J_{23}G^{\frac{1}{2}}\overline{v})_{i,j+\frac{1}{2},k+\frac{1}{2}} - (J_{23}G^{\frac{1}{2}}\overline{v})_{i,j+\frac{1}{2},k-\frac{1}{2}}}{\Delta z}, \tag{8.66}$$

$$(G^{\frac{1}{2}}S_{33})_{i,j,k} = \frac{J_{33}G^{\frac{1}{2}}\overline{w}_{i,j,k+\frac{1}{2}} - J_{33}G^{\frac{1}{2}}\overline{w}_{i,j,k-\frac{1}{2}}}{\Delta z}, \tag{8.67}$$

$$\begin{aligned}(G^{\frac{1}{2}}S_{12})_{i,j,k} = \frac{1}{2}\Bigg\{ &\frac{(G^{\frac{1}{2}}\overline{u})_{i,j+\frac{1}{2},k} - (G^{\frac{1}{2}}\overline{u})_{i,j-\frac{1}{2},k}}{\Delta y} + \frac{(G^{\frac{1}{2}}\overline{v})_{i+\frac{1}{2},j,k} - (G^{\frac{1}{2}}\overline{v})_{i-\frac{1}{2},j,k}}{\Delta x} \\ &+ \frac{(J_{23}G^{\frac{1}{2}}\overline{u})_{i,j,k+\frac{1}{2}} - (J_{23}G^{\frac{1}{2}}\overline{u})_{i,j,k-\frac{1}{2}} + (J_{13}G^{\frac{1}{2}}\overline{v})_{i,j,k+\frac{1}{2}} - (J_{13}G^{\frac{1}{2}}\overline{v})_{i,j,k-\frac{1}{2}}}{\Delta z}\Bigg\},\end{aligned} \tag{8.68}$$

$$\begin{aligned}(G^{\frac{1}{2}}S_{23})_{i,j,k} = \frac{1}{2}\Bigg\{ &\frac{(G^{\frac{1}{2}}\overline{w})_{i,j+\frac{1}{2},k} - (G^{\frac{1}{2}}\overline{w})_{i,j-\frac{1}{2},k}}{\Delta y} \\ &+ \frac{J_{33}G^{\frac{1}{2}}\overline{v}_{i,j,k+\frac{1}{2}} - J_{33}G^{\frac{1}{2}}\overline{v}_{i,j,k-\frac{1}{2}} + (J_{23}G^{\frac{1}{2}}\overline{w})_{i,j,k+\frac{1}{2}} - (J_{23}G^{\frac{1}{2}}\overline{w})_{i,j,k-\frac{1}{2}}}{\Delta z}\Bigg\},\end{aligned} \tag{8.69}$$

$$\begin{aligned}(G^{\frac{1}{2}}S_{31})_{i,j,k} = \frac{1}{2}\Bigg\{ &\frac{(G^{\frac{1}{2}}\overline{w})_{i+\frac{1}{2},j,k} - (G^{\frac{1}{2}}\overline{w})_{i-\frac{1}{2},j,k}}{\Delta x} \\ &+ \frac{J_{33}G^{\frac{1}{2}}\overline{u}_{i,j,k+\frac{1}{2}} - J_{33}G^{\frac{1}{2}}\overline{u}_{i,j,k-\frac{1}{2}} + (J_{13}G^{\frac{1}{2}}\overline{w})_{i,j,k+\frac{1}{2}} - (J_{13}G^{\frac{1}{2}}\overline{w})_{i,j,k-\frac{1}{2}}}{\Delta z}\Bigg\}.\end{aligned} \tag{8.70}$$

- z edge $(i+\frac{1}{2}, j+\frac{1}{2}, k)$

$$\begin{aligned}(G^{\frac{1}{2}}S_{12})_{i+\frac{1}{2},j+\frac{1}{2},k} = \frac{1}{2}\Bigg\{ &\frac{(G^{\frac{1}{2}}\overline{u})_{i+\frac{1}{2},j+1,k} - (G^{\frac{1}{2}}\overline{u})_{i+\frac{1}{2},j,k}}{\Delta y} + \frac{(G^{\frac{1}{2}}\overline{v})_{i+1,j+\frac{1}{2},k} - (G^{\frac{1}{2}}\overline{v})_{i,j+\frac{1}{2},k}}{\Delta x} \\ &+ \frac{(J_{23}G^{\frac{1}{2}}\overline{u})_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - (J_{23}G^{\frac{1}{2}}\overline{u})_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} + (J_{13}G^{\frac{1}{2}}\overline{v})_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - (J_{13}G^{\frac{1}{2}}\overline{v})}{\Delta z}\end{aligned} \tag{8.71}$$

- x edge $(i, j+\frac{1}{2}, k+\frac{1}{2})$

$$\begin{aligned}(G^{\frac{1}{2}}S_{23})_{i,j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{2}\Bigg\{ &\frac{(G^{\frac{1}{2}}\overline{w})_{i,j+1,k+\frac{1}{2}} - (G^{\frac{1}{2}}\overline{w})_{i,j,k+\frac{1}{2}}}{\Delta y} \\ &+ \frac{J_{33}G^{\frac{1}{2}}\overline{v}_{i,j+\frac{1}{2},k+1} - J_{33}G^{\frac{1}{2}}\overline{v}_{i,j+\frac{1}{2},k} + (J_{23}G^{\frac{1}{2}}\overline{w})_{i,j+\frac{1}{2},k+1} - (J_{23}G^{\frac{1}{2}}\overline{w})_{i,j+\frac{1}{2},k}}{\Delta z}.\Bigg\}\end{aligned} \tag{8.72}$$

- y edge $(i+\frac{1}{2}, j, k+\frac{1}{2})$

$$(G^{\frac{1}{2}} S_{31})_{i+\frac{1}{2},j,k+\frac{1}{2}} = \frac{1}{2}\left\{ \frac{(G^{\frac{1}{2}}\overline{w})_{i+1,j,k+\frac{1}{2}} - (G^{\frac{1}{2}}\overline{w})_{i,j,k+\frac{1}{2}}}{\Delta x} + \frac{J_{33}G^{\frac{1}{2}}\overline{u}_{i+\frac{1}{2},j,k+1} - J_{33}G^{\frac{1}{2}}\overline{u}_{i+\frac{1}{2},j,k} + (J_{13}G^{\frac{1}{2}}\overline{w})_{i+\frac{1}{2},j,k+1} - (J_{13}G^{\frac{1}{2}}\overline{w})_{i+\frac{1}{2},j,k}}{\Delta z}. \right\}. \tag{8.73}$$

velocity Calculation of the strain tensor requires velocity value at cell center, plane center, edge center, and vertex. The velocities at cell center (full-level) are eq.(3.87-3.89):

- x-y plane center $(i, j, k+\frac{1}{2})$

$$\overline{u}_{i,j,k+\frac{1}{2}} = \frac{\overline{u}_{i,j,k+1} + \overline{u}_{i,j,k}}{2}, \tag{8.74}$$

$$\overline{v}_{i,j,k+\frac{1}{2}} = \frac{\overline{v}_{i,j,k+1} + \overline{v}_{i,j,k}}{2}, \tag{8.75}$$

$$\overline{w}_{i,j,k+\frac{1}{2}} = \frac{(\rho w)_{i,j,k+\frac{1}{2}}}{\overline{\rho}_{i,j,k+\frac{1}{2}}}. \tag{8.76}$$

- y-z plane center $(i+\frac{1}{2}, j, k)$

$$\overline{u}_{i+\frac{1}{2},j,k} = \frac{(\rho u)_{i+\frac{1}{2},j,k}}{\overline{\rho}_{i+\frac{1}{2},j,k}}, \tag{8.77}$$

$$\overline{v}_{i+\frac{1}{2},j,k} = \frac{\overline{v}_{i+1,j,k} + \overline{v}_{i,j,k}}{2}, \tag{8.78}$$

$$\overline{w}_{i+\frac{1}{2},j,k} = \frac{\overline{w}_{i+1,j,k} + \overline{w}_{i,j,k}}{2}. \tag{8.79}$$

- z-x plane center $(i, j+\frac{1}{2}, k)$

$$\overline{u}_{i,j+\frac{1}{2},k} = \frac{\overline{u}_{i,j+1,k} + \overline{u}_{i,j,k}}{2}, \tag{8.80}$$

$$\overline{v}_{i,j+\frac{1}{2},k} = \frac{(\rho v)_{i,j+\frac{1}{2},k}}{\overline{\rho}_{i,j+\frac{1}{2},k}}, \tag{8.81}$$

$$\overline{w}_{i,j+\frac{1}{2},k} = \frac{\overline{w}_{i,j+1,k} + \overline{w}_{i,j,k}}{2}. \tag{8.82}$$

- x edge center $(i, j+\frac{1}{2}, k+\frac{1}{2})$

$$\overline{u}_{i,j+\frac{1}{2},k+\frac{1}{2}} = \frac{\overline{u}_{i,j+1,k+1} + \overline{u}_{i,j+1,k} + \overline{u}_{i,j,k+1} + \overline{u}_{i,j,k}}{4}, \tag{8.83}$$

$$\overline{v}_{i,j+\frac{1}{2},k+\frac{1}{2}} = \frac{\overline{v}_{i,j+\frac{1}{2},k+1} + \overline{v}_{i,j+\frac{1}{2},k}}{2}, \tag{8.84}$$

$$\overline{w}_{i,j+\frac{1}{2},k+\frac{1}{2}} = \frac{\overline{w}_{i,j+1,k+\frac{1}{2}} + \overline{w}_{i,j,k+\frac{1}{2}}}{2}. \tag{8.85}$$

- y edge center $(i+\frac{1}{2}, j, k+\frac{1}{2})$

$$\overline{u}_{i+\frac{1}{2},j,k+\frac{1}{2}} = \frac{\overline{u}_{i+\frac{1}{2},j,k+1} + \overline{u}_{i+\frac{1}{2},j,k}}{2}, \tag{8.86}$$

$$\overline{v}_{i+\frac{1}{2},j,k+\frac{1}{2}} = \frac{\overline{v}_{i+1,j,k+1} + \overline{v}_{i+1,j,k} + \overline{v}_{i,j,k+1} + \overline{v}_{i,j,k}}{4}, \tag{8.87}$$

$$\overline{w}_{i+\frac{1}{2},j,k+\frac{1}{2}} = \frac{\overline{w}_{i+1,j,k+\frac{1}{2}} + \overline{w}_{i,j,k+\frac{1}{2}}}{2}. \tag{8.88}$$

- z edge center $(i+\frac{1}{2}, j+\frac{1}{2}, k)$

$$\overline{u}_{i+\frac{1}{2},j+\frac{1}{2},k} = \frac{\overline{u}_{i+\frac{1}{2},j+1,k} + \overline{u}_{i+\frac{1}{2},j,k}}{2}, \tag{8.89}$$

$$\overline{v}_{i+\frac{1}{2},j+\frac{1}{2},k} = \frac{\overline{v}_{i+1,j+\frac{1}{2},k} + \overline{v}_{i,j+\frac{1}{2},k}}{2}, \tag{8.90}$$

$$\overline{w}_{i+\frac{1}{2},j+\frac{1}{2},k} = \frac{\overline{w}_{i+1,j+1,k} + \overline{w}_{i+1,j,k} + \overline{w}_{i,j+1,k} + \overline{w}_{i,j,k}}{4}. \tag{8.91}$$

- vertex $(i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2})$

$$\overline{u}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = \frac{\overline{u}_{i+\frac{1}{2},j+1,k+1} + \overline{u}_{i+\frac{1}{2},j+1,k} + \overline{u}_{i+\frac{1}{2},j,k+1} + \overline{u}_{i+\frac{1}{2},j,k}}{4}, \tag{8.92}$$

$$\overline{v}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = \frac{\overline{v}_{i+1,j+\frac{1}{2},k+1} + \overline{v}_{i+1,j+\frac{1}{2},k} + \overline{v}_{i,j+\frac{1}{2},k+1} + \overline{v}_{i,j+\frac{1}{2},k}}{4}, \tag{8.93}$$

$$\overline{w}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = \frac{\overline{w}_{i+1,j+1,k+\frac{1}{2}} + \overline{w}_{i+1,j,k+\frac{1}{2}} + \overline{w}_{i,j+1,k+\frac{1}{2}} + \overline{w}_{i,j,k+\frac{1}{2}}}{4}. \tag{8.94}$$

Eddy viscosity/diffusion coefficient The eddy viscosity/diffusion coefficient, ν_{SGS} / D_{SGS}, is calculated at full-level with S and Ri at full-level; at half-level, it is interpolated to full-level.

Brunt-Visala frequency The Brunt-Visala frequency, N^2 is required to calculate the Richardson number at full-level.

$$(G^{\frac{1}{2}} N^2)_{i,j,k} = \frac{g}{\theta_{i,j,k}} \frac{J_{33} G^{\frac{1}{2}} \theta_{i,j,k+1} - J_{33} G^{\frac{1}{2}} \theta_{i,j,k-1}}{2\Delta z}. \tag{8.95}$$

8.2 Boundary layer turbulence model

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8.2.1 Mellor-Yamada Nakanishi-Niino model

level 2.5

$$\frac{\partial \rho u}{\partial t} = -\frac{\partial}{\partial z}\rho\overline{u'w'}, \tag{8.96}$$

$$\frac{\partial \rho v}{\partial t} = -\frac{\partial}{\partial z}\rho\overline{v'w'}, \tag{8.97}$$

$$\frac{\partial \rho \theta_l}{\partial t} = -\frac{\partial}{\partial z}\rho\overline{\theta_l' w'}, \tag{8.98}$$

$$\frac{\partial \rho q_a}{\partial t} = -\frac{\partial}{\partial z}\rho\overline{q_a' w'}, \tag{8.99}$$

$$\frac{\partial}{\partial t}\rho q^2 = -2\left(\rho\overline{u'w'}\frac{\partial u}{\partial z} + \rho\overline{v'w'}\frac{\partial v}{\partial z}\right) + 2\frac{g}{\theta_0}\rho\overline{\theta_v' w'} - \frac{\partial}{\partial z}\rho\overline{q^2 w'} - 2\rho\epsilon, \tag{8.100}$$

where:

$$q_a = q_v + q_c + q_r + q_i + q_s + q_g, \tag{8.101}$$

and q^2 is doubled turbulence kinetic energy:

$$q^2 = u'^2 + v'^2 + w'^2. \tag{8.102}$$

The higher order moments and the dissipation term are parameterized as follows:

$$\overline{u'w'} = -LqS_M\frac{\partial u}{\partial z}, \tag{8.103}$$

$$\overline{v'w'} = -LqS_M\frac{\partial v}{\partial z}, \tag{8.104}$$

$$\overline{\theta_l' w'} = -LqS_H\frac{\partial \theta_l}{\partial z}, \tag{8.105}$$

$$\overline{q_a' w'} = -LqS_H\frac{\partial q_a}{\partial z}, \tag{8.106}$$

$$\overline{q^2 w'} = -3LqS_M\frac{\partial q^2}{\partial z}, \tag{8.107}$$

$$\overline{\theta_v' w'} = \beta_\theta\overline{\theta_l' w}' + \beta_q\overline{q_a' w'}, \tag{8.108}$$

$$\epsilon = \frac{q^3}{B_1 L}, \tag{8.109}$$

where:

$$S_M = \alpha_c A_1\frac{\Phi_3 - 3C_1\Phi_4}{D_{2.5}}, \tag{8.110}$$

$$S_H = \alpha_c A_2\frac{\Phi_2 + 3C_1\Phi_5}{D_{2.5}}, \tag{8.111}$$

$$\beta_\theta = 1 + 0.61q_a - 1.61Q_l - \tilde{R}abc, \tag{8.112}$$

$$\beta_q = 0.61\theta + \tilde{R}ac. \tag{8.113}$$

$$D_{2.5} = \Phi_2\Phi_4 + \Phi_5\Phi_3, \tag{8.114}$$

$$\Phi_1 = 1 - 3\alpha_c^2 A_2 B_2 (1 - C_3) G_H, \tag{8.115}$$

$$\Phi_2 = 1 - 9\alpha_c^2 A_1 A_2 (1 - C_2) G_H, \tag{8.116}$$

$$\Phi_3 = \Phi_1 + 9\alpha_c^2 A_2^2 (1 - C_2)(1 - C_5) G_H, \tag{8.117}$$

$$\Phi_4 = \Phi_1 - 12\alpha_c^2 A_1 A_2 (1 - C_2) G_H, \tag{8.118}$$

$$\Phi_5 = 6\alpha_c^2 A_1^2 G_M, \tag{8.119}$$

$$\alpha_c = \begin{cases} q/q_2, & q < q_2 \\ 1, & q \geq q_2 \end{cases}, \tag{8.120}$$

$$G_M = \frac{L^2}{q^2}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}, \tag{8.121}$$

$$G_H = -\frac{L^2}{q^2} N^2, \tag{8.122}$$

$$R = \frac{1}{2}\left\{1 + \mathrm{erf}\left(\frac{Q_1}{\sqrt{2}}\right)\right\}, \tag{8.123}$$

$$\tilde{R} = R - \frac{Q_l}{2\sigma_s}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{q_1^2}{2}\right), \tag{8.124}$$

$$Q_l = 2\sigma_s\left\{RQ_1 + \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{Q_1^2}{2}\right)\right\}, \tag{8.125}$$

$$Q_1 = \frac{a}{2\sigma_s}(q_a - Q_{sl}), \tag{8.126}$$

$$\sigma_s^2 = \frac{1}{4}a^2 L^2 \alpha_c B_2 S_H \left(\frac{\partial q_a}{\partial z} - b\frac{\partial \theta_l}{\partial z}\right)^2, \tag{8.127}$$

$$\delta Q_{sl} = \left.\frac{\partial Q_s}{\partial T}\right|_{T=T_l}, \tag{8.128}$$

$$a = \left(1 + \frac{L}{C_p}\delta Q_{sl}\right)^{-1}, \tag{8.129}$$

$$b = \frac{T}{\theta}\delta Q_{sl}, \tag{8.130}$$

$$c = (1 + 0.61q_a - 1.61Q_l)\frac{\theta}{T}\frac{L_v}{C_p} - 1.61\theta, \tag{8.131}$$

and Q_{sl} is the saturation-specific humidity at temperature $T_l (= \theta_l T/\theta)$.

The buoyancy flux term, which is the third term on the left hand side of eq. 8.100 is:

$$\begin{aligned} 2\frac{g}{\theta_0}\overline{\theta_v' w'} &= 2\frac{g}{\theta_0}\left(-\beta_\theta L q S_H \frac{\partial \theta_l}{\partial z} - \beta_q L q S_H \frac{\partial q_a}{\partial z}\right) \\ &= -2LqS_H \frac{g}{\theta_0}\left(\beta_\theta \frac{\partial \theta_l}{\partial z} + \beta_q \frac{\partial q_a}{\partial z}\right) \\ &= -2LqS_H \frac{g}{\theta_0}\frac{\partial \theta_v}{\partial z} \\ &= -2LqS_H N^2, \end{aligned} \tag{8.132}$$

where N^2 is the square of the Brunt-Vaisala frequency.

$$\frac{\partial}{\partial t}\rho q^2 = 2\rho L q S_M \left\{ \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right\} \\ - 2\rho L q S_H N^2 + \frac{\partial}{\partial z}\left(3\rho L q S_M \frac{\partial}{\partial z} q^2\right) - 2\rho \frac{q^3}{B_1 L} \quad (8.133)$$

S_{M2}, S_{H2}, and q_2 are for level 2 schemes corresponding to S_M, S_H, and q, respectively:

$$S_{M2} = \frac{A_1 F_1}{A_2 F_2} \frac{R_{f1} - Rf}{R_{f2} - Rf} S_{H2}, \quad (8.134)$$

$$S_{H2} = 3A_2(\gamma_1 + \gamma_2) \frac{Rf_c - Rf}{1 - Rf}, \quad (8.135)$$

$$q_2^2 = B_1 L^2 S_{M2}(1 - Rf) \left\{ \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right\}. \quad (8.136)$$

Rf and Rf_c are the flux Richardson number and the critical flux Richardson number, respectively. The gradient Richardson number, Ri, is:

$$Ri = Rf \frac{S_{M2}}{S_{H2}}. \quad (8.137)$$

Rf is then:

$$Rf = \frac{1}{2}\frac{A_2 F_2}{A_1 F_1} \left\{ Ri + \frac{A_1 F_1}{A_2 F_2} R_{f1} - \sqrt{Ri^2 + 2\frac{A_1 F_1}{A_2 F_2}(R_{f1} - 2R_{f2}) Ri + \left(\frac{A_1 F_1}{A_2 F_2} R_{f1}\right)^2} \right\}, \quad (8.138)$$

$$Rf_C = \frac{\gamma_1}{\gamma_1 + \gamma_2}, \quad (8.139)$$

$$(8.140)$$

where:

$$R_{f1} = B_1 \frac{\gamma_1 - C_1}{F_1}, \quad (8.141)$$

$$R_{f2} = B_1 \frac{\gamma_1}{F_2}. \quad (8.142)$$

The turbulent length scale, L, is determined by the smallest length scale among three scales:

$$\frac{1}{L} = \frac{1}{L_s} + \frac{1}{L_T} + \frac{1}{L_B}. \quad (8.143)$$

the surface layer scale, L_s, the boundary layer scale, L_T, and buoyancy length scale, T_B:

$$L_S = \begin{cases} kz/3.7, & \zeta \geq 1 \\ kz/(1+2.7\zeta), & 0 \leq \zeta < 1 \\ kz(1-100\zeta)^{0.2}, & \zeta < 0 \end{cases}, \tag{8.144}$$

$$L_T = 0.23 \frac{\int_0^\infty qz dz}{\int_0^\infty q dz}, \tag{8.145}$$

$$L_B = \begin{cases} q/N, & \partial\theta_v/\partial z > 0 \text{ and } \zeta \geq 0 \\ \{1 + 5(q_c/L_T N)^{1/2}\} q/N, & \partial\theta_v/\partial z > 0 \text{ and } \zeta < 0 \\ \infty, & \partial\theta_v/\partial z \leq 0 \end{cases}, \tag{8.146}$$

where ζ is the dimensionless height:

$$\zeta = \frac{z}{L_M}. \tag{8.147}$$

L_M is the Monin-Obukhov length:

$$L_M = -\frac{\theta_0 u_*^3}{kg\overline{\theta_v' w'}_g}, \tag{8.148}$$

where u_* is the friction velocity, and the subscript g denotes the ground surface. q_c is a velocity scale defined in a similar manner to convective velocity w_*, except that the depth z_i of the convective boundary layer is replaced by L_t:

$$q_c = \left\{ \frac{g}{\theta_0} \overline{\theta_v' w'}_g L_T \right\}^{1/3} \tag{8.149}$$

$$A_1 = B_1 \frac{1 - 3\gamma_1}{6}, \tag{8.150}$$

$$A_2 = \frac{1}{3\gamma_1 B_1^{1/3} Pr_N}, \tag{8.151}$$

$$B_1 = 24.0, \tag{8.152}$$

$$B_2 = 15.0, \tag{8.153}$$

$$C_1 = \gamma_1 - \frac{1}{3A_1 B_1^{1/3}}, \tag{8.154}$$

$$C_2 = 0.75, \tag{8.155}$$

$$C_3 = 0.352, \tag{8.156}$$

$$C_5 = 0.2, \tag{8.157}$$

$$\gamma_1 = 0.235, \tag{8.158}$$

$$\gamma_2 = \frac{2A_1(3 - 2C_2) + B_2(1 - C_3)}{B_1}, \tag{8.159}$$

$$F_1 = B_1(\gamma_1 - C_1) + 2A_1(3 - 2C_2) + 3A_2(1 - C_2)(1 - C_5), \tag{8.160}$$

$$F_2 = B_1(\gamma_1 + \gamma_2) - 3A_1(1 - C_2), \tag{8.161}$$

$$Pr_N = 0.74. \tag{8.162}$$

Discretization

The diffusion equations for q^2a are solved implicitly:

$$\rho_k \frac{(q_k^2)^{n+1} - (q_k^2)^n}{\Delta t} = 2\rho_k \left[(LqS_M)_k \left\{ \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right\} + (LqS_H N^2)_k \right]$$
$$+ \frac{1}{\Delta z_k} \left\{ (3\rho LqS_M)_{k+\frac{1}{2}} \frac{(q_{k+1}^2)^{n+1} - (q_k^2)^{n+1}}{\Delta z_{k+\frac{1}{2}}} - (3\rho LqS_M)_{k-\frac{1}{2}} \frac{(q_k^2)^{n+1} - (q_{k-1}^2)^{n+1}}{\Delta z_{k-\frac{1}{2}}} \right\}$$
$$- \frac{2\rho_k q_k}{B_1 L_k} (q_k^2)^{n+1}. \tag{8.163}$$

$$a_k (q_{k+1}^2)^{n+1} + b_k (q_k^2)^{n+1} + c_k (q_{k-1}^2)^{n+1} = d_k, \tag{8.164}$$

where:

$$a_k = -\frac{\Delta t}{\Delta z_{k+\frac{1}{2}} \Delta z_k \rho_k} (3\rho LqS_M)_{k+\frac{1}{2}}, \tag{8.165}$$

$$b_k = -a_k - c_k + 1 + \frac{2\Delta t q_k}{B_1 L}, \tag{8.166}$$

$$c_k = -\frac{\Delta t}{\Delta z_k \Delta z_{k-\frac{1}{2}} \rho_k} (3\rho LqS_M)_{k-\frac{1}{2}}, \tag{8.167}$$

$$d_k = (q_k^2)^n + 2\Delta t \left[LqS_M \left\{ \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right\} - LqS_H N^2 \right] \tag{8.168}$$

$$(q_k^2)^{n+1} = e_k (q_{k+1}^2)^{n+1} + f_k, \tag{8.169}$$

where:

$$e_k = -\frac{a_k}{b_k + c_k e_{k-1}}, \tag{8.170}$$

$$f_k = \frac{d_k - c_k f_{k-1}}{b_k + c_k e_{k-1}}. \tag{8.171}$$

Vertical fluxes for $\rho u, \rho v, \rho\theta, \rho q_x$ are also solved implicitly. For instance, the flux for ρu, F_u is calculated by:

$$F_{u,k+\frac{1}{2}} = (\rho LqSM)_{k+\frac{1}{2}} \frac{u_{k+1}^{n+1} - u_k^{n+1}}{\Delta z_{k+\frac{1}{2}}}. \tag{8.172}$$

u^{n+1} is calculated as the same way with q^2, but:

$$a_k = -\frac{\Delta t}{\Delta z_{k+\frac{1}{2}} \Delta z_k \rho_k} (\rho LqS_M)_{k+\frac{1}{2}}, \tag{8.173}$$

$$b_k = -a_k - c_k + 1, \tag{8.174}$$

$$c_k = -\frac{\Delta t}{\Delta z_k \Delta z_{k-\frac{1}{2}} \rho_k} (\rho LqS_M)_{k-\frac{1}{2}}, \tag{8.175}$$

$$d_k = u_k^n. \tag{8.176}$$

8.3 Microphysics

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SCALE-RM has three types of cloud microphysics models. We provide a description of these models below.

8.3.1 Kessler Parameterization

SCALE implements a one-moment bulk microphysical scheme, which treats only warm clouds (cloud and rain). This scheme predicts the mixing ratio of cloud (Q_{cloud}) and rain (Q_{rain}). Cloud microphysical processes treated in this scheme are saturation adjustment (corresponding to nucleation, evaporation, and cloud condensation), evaporation, auto-conversion, accretion, and sedimentation. The tendency of Q_{cloud}, Q_{rain}, and Q_v (vapor mixing ratio) is as follows:

$$\frac{\partial Q_{cloud}}{\partial t} = dQ|_{sat} - dQ|_{auto} - dQ|_{acc} \quad (8.177)$$

$$\frac{\partial Q_{rain}}{\partial t} = dQ|_{auto} + dQ|_{acc} - dQ|_{evap} - F_{Q_r}|_{sed} \quad (8.178)$$

$$\frac{\partial Q_v}{\partial t} = dQ|_{evap} - dQ|_{sat} \quad (8.179)$$

where $dQ|_{sat}$, $dQ|_{auto}$, $dQ|_{acc}$, and $dQ|_{evap}$ represent the mixing ratio tendency by saturation adjustment, auto-conversion, accretion, and evaporation, respectively. $F_{Q_r}|_{sed}$ represents flux of Q_r by sedimentation.
$dQ|_{auto}$, $dQ|_{acc}$, and $dQ|_{evap}$ are given as:

$$dQ_{auto} = \begin{cases} Q_{cloud} * 10^{-3} & (Q_{cloud} > 10^{-3}) \\ 0 & (else) \end{cases} \quad (8.180)$$

$$dQ_{acc} = 2.2 \times Q_{cloud} \times Q_{rain}^{0.875} \quad (8.181)$$

$$dQ_{evap} = \begin{cases} f_{vent} \frac{q_s - Q_{cloud}}{q_s \rho} \frac{(\rho * Q_{rain})^{0.525}}{5.4 \times 10^5 + \frac{2.55 \times 10^8}{p q_s}} & (q_s > Q_{cloud}) \\ 0 & (else) \end{cases} \quad (8.182)$$

where f_{vent} is the ventilation factor ($f_{vent} = 1.6 + 124.9(\rho Q_{rain})^{0.2046}$), and the unit of dQ_{***} is [kg/kg/s]. p, q_s, and ρ are pressure, saturation vapor mixing ratio, and total density, respectively.
$dQ|_{sat}$ is given as:

$$dQ|_{sat} = Q_v - q_s. \quad (8.183)$$

Terminal velocities of cloud ($V_{t,c}$) and rain ($V_{t,r}$) are given as:

$$V_{t,c} = 0 \quad (8.184)$$

$$V_{t,r} = 36.34(\rho Q_{rain})^{0.1364}[m/s] \quad (8.185)$$

8.3.2 Spectral Bin Model(SBM)

The Spectral Bin Model (SBM) was developed by Suzuki (2006) and Suzuki et al. (2010). The model forecasts the Size Distribution Function (SDF) of seven types of hydrometeors (liquid, plate-ice, columnar-ice, dendritic-ice, snow, graupel, and hail).
The SBM calculates mass density of the seven types of hydrometeor and one type of aerosol as their SDFs. The SDF of aerosol can be changed by advection and activation (i.e., nucleation from aerosol to cloud) processes. The SDF of hydrometeors can be changed by several growth processes (i.e., activation from aerosol to cloud, condensation/evaporation, collision/coagulation, freezing/melting, ice nucleation, riming, aggregation, advection, and gravitational falling).
The time evolution of SDF (number density) of aerosol ($f_a(m,t)$) and SDF (number density) of hydrometeor ($f_c(m,t)$) are shown as:

$$\frac{\partial f_c^{(\mu)}(m,t)}{\partial t} = Adv\left[f_c^{(\mu)}(m,t)\right] + Grav\left[f_c^{(\mu)}(m,t)\right] + \left[\frac{\partial f_c^{(\mu)}(m,t)}{\partial t}\right]_{cloud\ microphysics} \quad (8.186)$$

$$\frac{\partial f_a(m_a,t)}{\partial t} = Adv\left[f_a(m_a,t)\right] + Grav\left[f_a(m_a,t)\right] + \left[\frac{\partial f_a(m_a,t)}{\partial t}\right]_{cloud\ microphysics} \quad (8.187)$$

where μ shows type of hydrometeor (the seven types), and $Adv[]$, $Grav[]$) show change of SDF by advection and gravitational falling. $\Big[\Big]_{cloud\ microphysics}$ shows SDF changes by cloud microphysical processes.
The time evolution of $f_c^{(\mu)}(m,t)$, and $f_a(m,t)$ are shown as:

$$\begin{aligned}
\left[\frac{\partial f_c^{(\mu)}(m,t)}{\partial t}\right]_{cloud\ microphysics} &= \left[\frac{\partial f_c^{(\mu)}(m,t)}{\partial t}\right]_{activation} + \left[\frac{\partial f_c^{(\mu)}(m,t)}{\partial t}\right]_{cond/evap} \\
&+ \left[\frac{\partial f_c^{(\mu)}(m,t)}{\partial t}\right]_{coll/coag/rim/agg} \\
&+ \left[\frac{\partial f_c^{(\mu)}(m,t)}{\partial t}\right]_{frz} + \left[\frac{\partial f_c^{(\mu)}(m,t)}{\partial t}\right]_{melt} \\
\left[\frac{\partial f_a(m_a,t)}{\partial t}\right]_{cloud\ microphysics} &= \left[\frac{\partial f_a(m_a,t)}{\partial t}\right]_{activation}
\end{aligned}$$

where $\Big[\Big]_{***}$ show change of SDF by each cloud growth process. The detail of these processes will be provided later.
The change of SDFs by advection and gravitational falling (i.e., first and second terms of eq.(8.186), and (8.187)) are calculated by dynamical core of SCALE-RM shown in section 3.

Discretization of Size Distribution Function(SDF)

The SDF of aerosol and cloud is predicted as mass density of each particle size ($g_a(m_a)$, $g_c^{(\mu)}(m)$). However most equations are given as equations of number density of cloud/aerosol ($f_c^{(\mu)}(m,t)$, $f_a(m_a,t)$); the mass density of cloud/aerosol is transferred to the number density of cloud/aerosol ($g_a(m_a,t) =$

$m_a g_a(m_a,t)$, $g_c^{(\mu)}(m,t) = m^{(\mu)} f_c^{(\mu)}(m,t)$).
To cover a wide size range (i.e., 2 $\mu m \sim 3$ mm), a logarithmically uniform grid system ($log(m) \equiv \eta$, $log(m_a) \equiv \eta_a$) is used. In this system, the relationship, $\frac{m_{i+1}}{m_i} = const.$ is satisfied.

Activation from aerosol to cloud particles (nucleation process)

The change of SDFs by activation from aerosol to cloud particles is calculated based on Kohler theory Kohler (1936). Through this process, aerosols with radii larger than the aerosol critical radius ($r_{a,crit}$) are activated to clouds. The critical radius is given as:

$$r_{a,crit} = \Big(\frac{4}{27}\frac{A^3}{B}\frac{1}{S_w}\Big)^{1/3}, \quad A = \frac{2\sigma}{R_v \rho_L T}, \quad B = i_v \frac{M_v}{M_s}\frac{\rho_s}{\rho_L}. \tag{8.188}$$

where S_w, σ, R_v, ρ_L, T, i_v, M_v, M_s, and ρ_s show supersaturation of water, surface tension of water, vapor gas constant, temperature, van't Hoff factor (= 2), molecular weight of water, molecular weight of aerosol, and density of aerosol, respectively.
At each time step, $r_{a,crit}$ is calculated using temperature, and masses of aerosols with radii ¿ $r_{a,crit}$ are removed from SDF of aerosol and transferred to SDF of cloud as newly generated cloud particles.
The radii of newly generated clouds correspond to those of aerosols, but if the radii of aerosols are smaller than the lower limit of cloud SDF, the radii of newly generated clouds are set to the smallest size of cloud SDF ($\sim 2\mu m$).
The changes in aerosol and hydrometeor SDF are shown as:

$$\Big[\frac{\partial f_a}{\partial t}\Big]_{activation} = -\int_{m_{a,crit}}^{\infty} f_a(m_a,t)dm_a \tag{8.189}$$

$$\Big[\frac{\partial f_c^{(\mu)}}{\partial t}\Big]_{activation} = -\Big[\frac{\partial f_a}{\partial t}\Big]_{activation} \tag{8.190}$$

where $m_{a,crit} = (= \frac{4\pi}{3} r_a^3 \rho_a$ is mass of aerosol particles with radii the same as critical radii, $r_{a,crit}$. When there is not enough vapor to activate all aerosol particles with radii larger than the critical radius, i.e.,

$$\int_{m_{a,crit}}^{\infty} m_a f_a(m_a,t)dm_a > q_v \rho, \tag{8.191}$$

only the aerosol particles with radii ¿ than $r_{a0,crit}$, given as:

$$\int_{m_{a0,crit}}^{\infty} m_a f_a(m_a,t)dm_a = q_v \rho, \tag{8.192}$$

are transferred to cloud particles as:

$$\Big[\frac{\partial f_a}{\partial t}\Big]_{activation} = -\int_{m_{a0,crit}}^{\infty} f_a(m_a,t)dm_a, \tag{8.193}$$

$$\Big[\frac{\partial f_c^{(\mu)}}{\partial t}\Big]_{activation} = -\Big[\frac{\partial f_a}{\partial t}\Big]_{activation}. \tag{8.194}$$

where q_v and ρ is the mixing ratio of water vapor and density.

Condensation/evaporation

Calculation of condensation and evaporation processes is based on an equation. The mass change by these two process is given by an equation (e.g., Rogers and Yau (1989)):

$$\frac{dm}{dt} = C^{(\mu)}(m)G^{(\mu)}(T)S^{(\mu)} \tag{8.195}$$

$$G^{(\mu)}(T) = \begin{cases} G_w(T) & (\mu:\ liquid) \\ G_i(T) & (\mu:\ ice) \end{cases}$$

$$G_w(T) = \frac{4\pi}{\frac{R_vT}{e_w(T)D_v} + \frac{L_w}{KT}\left(\frac{L_w}{R_vT} - 1\right)}$$

$$G_i(T) = \frac{4\pi}{\frac{R_vT}{e_i(T)D_v} + \frac{L_i}{KT}\left(\frac{L_i}{R_vT} - 1\right)}$$

$$S^{(\mu)} = \begin{cases} S_w & (\mu : liquid) \\ S_i & (\mu : ice) \end{cases}$$

where $C^{(\mu)}(m)$ is capacitance, which depends on the shape of each type of hydrometeor, S_w, S_i are super saturation of water and ice, L_w, L_i are sensible heat of evaporation, sublimation, D_v is diffusion constant of vapor, K is conductivity of air, and e_w, e_i are saturation vapor pressure and saturation ice pressure, respectively. Condensation (evaporation) occur when $S^{(\mu)}$ is positive (negative).

To calculate change of SDF by condensation/evaporation, mass flux ($F^{(\mu)}_{cond/evap}$) on each bin is given by using number density ($f^{(\mu)}_c$) and $\frac{dm}{dt}$ as:

$$F^{(\mu)}_{cond/evap} = f^{(\mu)}(m)\frac{dm}{dt} = f^{(\mu)}(m)C^{(\mu)}G^{(\mu)}(T)S^{(\mu)}. \tag{8.196}$$

Using this equation, time evolution of SDF ($f^{(\mu)}$) is given as

$$\begin{aligned} \left[\frac{\partial f^{(\mu)}(m,t)}{\partial t}\right]_{cond/evap} &= -\frac{\partial}{\partial m}F^{(\mu)}_{cond/evap}(m) \\ &= -\frac{\partial}{\partial m}\left(f^{(\mu)}(m)C^{(\mu)}\right)G^{(\mu)}(T)S^{(\mu)}. \end{aligned} \tag{8.197}$$

By using the $\eta(= \log(m))$, eq.(8.197) is transferred to the advection equation:

$$\frac{\partial f^{(\mu)}(\eta)}{\partial t} = -\frac{\partial}{\partial \eta}\left(f^{(\mu)}(\eta)U^{(\mu)}(\eta)\right) \tag{8.198}$$

$$U^{(\mu)}(\eta) = \frac{C^{(\mu)}(\eta)}{\exp(\eta)}G^{(\eta)}(T)S^{(\eta)}.$$

To solve eq.(8.198), a scheme developed by Bott (1989) is used. The number density of the i-th bin after Δt ($f_i(t + \Delta t)$) is given as follows:

$$
\begin{aligned}
f_i(t+\Delta t) &= f_i(t) - \frac{\Delta t}{\Delta \eta}\Big[F_{cond/evap,i+1/2} - F_{cond/evap,i-1/2}\Big]. \\
F_{cond/evap,i+1/2} &= \frac{\Delta \eta}{\Delta t}\Big[\frac{i^{+}_{l,i+1/2}}{i_{l,j}} f_i(t) - \frac{i^{-}_{l,i+1/2}}{i_{l,i+1}} f_{i+1}(t)\Big] \\
i^{+}_{l,i+1/2} &= max\big(0, I^{+}_{l}(c_{i+1/2})\big) \\
i^{-}_{l,i+1/2} &= max\big(0, I^{-}_{l}(c_{i+1/2})\big) \\
i^{+}_{l,i} &= max\big(I_{l,i}, i^{+}_{l,i+1/2} + i^{-}_{l,i+1/2})\big) \\
I^{+}_{l}(c_{i+1/2}) &= \sum_{k=0}^{2} \frac{a_{i,k}}{(k+1)2^{k+1}}\big[1-(1-2c^{+}_{j})^{k+1}\big] \\
I^{-}_{l}(c_{i+1/2}) &= \sum_{k=0}^{2} \frac{a_{i+1,k}}{(k+1)2^{k+1}}(-1)^{k}\big[1-(1-2c^{-}_{j})^{k+1}\big] \\
a_{i,0} &= -\frac{1}{24}\big(f_{i+1}(t) - 26 f_i(t) + f_{i-1}(t)\big) \\
a_{i,1} &= \frac{1}{2}\big(f_{i+1}(t) - f_{i-1}(t)\big) \\
a_{i,2} &= \frac{1}{2}\big(f_{i+1}(t) - 2f_i(t) + f_{i-1}(t)\big) \\
c^{\pm}_{i} &= \pm\big(c^{n}_{i+1/2} \pm |c^{n}_{i+1/2}|\big)/2 \\
c^{n}_{i+1/2} &= U^{n}_{i+1/2}\frac{\Delta t}{\Delta \eta} \qquad (8.199)
\end{aligned}
$$

Since super saturation ($S^{(\mu)}$) can change during time step (Δt), we apply a method shown below to reflect the change of supersaturation during Δt.
Time evolution of supersaturation can be given by equations:

$$
\begin{aligned}
\frac{d}{dt}\begin{pmatrix} S_w \\ S_i \end{pmatrix} &= \begin{pmatrix} a_{c/e} & b_{c/e} \\ c_{c/e} & d_{c/e} \end{pmatrix}\begin{pmatrix} S_w \\ S_i \end{pmatrix} = A\begin{pmatrix} S_w \\ S_i \end{pmatrix} \qquad (8.200) \\
a_{c/e} &= -(S_w+1)\Big(\frac{1}{q_v} + \frac{L_w}{R_v T^2}\frac{L_w}{C_p}\Big)\int f^{(w)}(m) C^{(w)}(m) dm G_w(t) \\
b_{c/e} &= -(S_w+1)\Big(\frac{1}{q_v} + \frac{L_w}{R_v T^2}\frac{L_i}{C_p}\Big)\sum_{\mu\in ice}\int f^{(\mu)}(m) C^{(\mu)}(m) dm G_i(t) \\
c_{c/e} &= -(S_i+1)\Big(\frac{1}{q_v} + \frac{L_i}{R_v T^2}\frac{L_w}{C_p}\Big)\int f^{(w)}(m) C^{(w)}(m) dm G_w(t) \\
d_{c/e} &= -(S_i+1)\Big(\frac{1}{q_v} + \frac{L_i}{R_v T^2}\frac{L_i}{C_p}\Big)\sum_{\mu\in ice}\int f^{(\mu)}(m) C^{(mu)}(m) dm G_i(t)
\end{aligned}
$$

where q_v is the mixing ratio of vapor.
Using eigen value of A (Λ_+, Λ_- ($\Lambda_+ > \Lambda_-$)), and assuming $a_{c/e}$, $b_{c/e}$, $c_{c/e}$, $d_{c/e}$ are constant during Δt, average value of super saturation ($\bar{S}_{w,i}(t)$) during Δt is given as:

$$\bar{S}_w(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} S_w(\tau)d\tau = b\frac{e^{\Lambda_+\Delta t}-1}{\Lambda_+\Delta t}S_+(t) + b\frac{e^{\Lambda_-\Delta t}-1}{\Lambda_-\Delta t}S_-(t)$$

$$\bar{S}_i(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} S_i(\tau)d\tau = (\Lambda_+ - a)\frac{e^{\Lambda_+\Delta t}-1}{\Lambda_+\Delta t}S_+(t) + (\Lambda_- - a)\frac{e^{\Lambda_-\Delta t}-1}{\Lambda_-\Delta t}S_-(t)$$

$$S_+(t) = \frac{(\Lambda_- - a)S_w(t) - bS_i(t)}{b(\Lambda_- - \Lambda_+)}$$

$$S_+(t) = \frac{(a - \Lambda_+)S_w(t) + bS_i(t)}{b(\Lambda_- - \Lambda_+)}$$

The averaged super saturation ($\bar{S}_{w,i}(t)$) is used to solve the eq.(8.200).

Collision/coagulation/riming/aggregation

Collision/coagulation processes are calculated by solving stochastic collision equations (e.g., Pruppacher and Klett (1997)):

$$\frac{\partial f(m)}{\partial t} = \int_0^{m/2} f(m')f(m-m')K(m',m-m')dm' - f(m)\int_0^{\infty} f(m'')K(m,m'')dm'' \quad (8.201)$$

where $K(m,m')$ is collection kernel function. Three types of kernel function, i.e., Long type kernel (Long (1974)), Golovin type kernel (Golovin (1963)), and Hydro-dynamic dynamic kernel as shown in eq.(8.202), are implemented into the SCALE-RM.

$$K(m,m') = \pi(r(m) - r(m'))\left|V(m) - V(m')\right| E_{col}(m,m')E_{coag}(m,m') \quad (8.202)$$

where $r(m)$ is radius of hydrometeors with mass m and $V(m)$ is terminal velocity of hydrometeors. The terminal velocity of each species of hydrometeor and each size are shown in Figure 8.1 E_{col}, and E_{coag} are collision efficiency and coagulation efficiency, respectively.

Although the stochastic collision equation can be applied for collision/coagulation of one type of hydrometeor (i.e., liquid water), SCALE-RM predicts seven types of hydrometeors and interactions of these types (i.e., riming/aggregation) must be calculated. To calculate the interaction of all seven types of hydrometeors, the extended stochastic collision equation:

$$\left[\frac{\partial f^{(\mu)}(m)}{\partial t}\right]_{coll/coag/rim/agg} = \sum_\lambda\sum_\nu\int_0^{m/2} f^{(\lambda)}(m')f^{(\nu)}(m-m')K_{\lambda\nu}(m',m-m')dm' - f^{(\mu)}(m)\sum_\kappa\int_0^{\infty} f^{(\kappa)}(m'')K_{\kappa\mu}(m,m'')dm'' \quad (8.203)$$

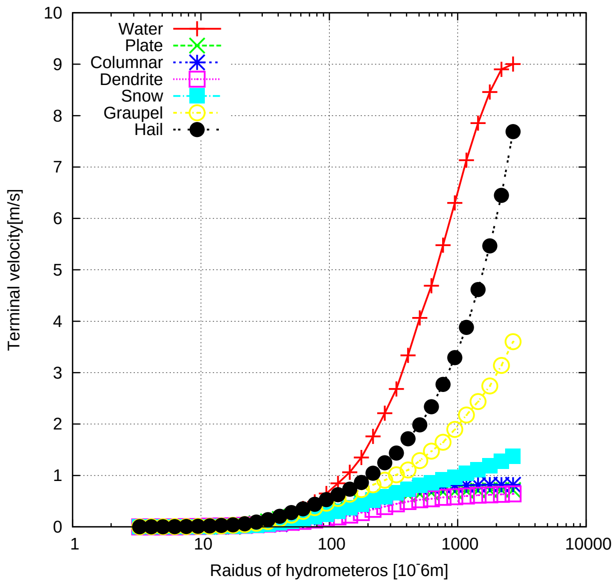

Figure 8.1: Terminal velocity of Water (Plus), plate-type ice (cross), columnar-type ice (asterisk), dendritic-type ice (open square), snow (closed square), graupel (open circle), and hail (closed circle). Cited from Figure A3 of Suzuki (2006) and rearranged.

is applied (where μ, ν, λ, κ represent species of hydrometeor). The combinations of μ, ν, λ are shown in table 8.1.

Table 8.1: Catalog of interaction between seven species. W, I, S, G, and H show water, ice, snow, graupel, and hail, respectively. G/H shows graupel(hail) generated when T is lower(higher) than 270.15 K

	W	I	S	G	H
W	W	G/H	G/H	G/H	G/H
I	I	S	S	I	I
S	S	S	S	S	S
G	G/H	G/H	G	G	G/H
H	G/H	G/H	G/H	G/H	H

To solve the stochastic collision equation, a scheme developed by Bott (1998) was implemented into SCALE-RM.
The Bott (1998) scheme calculates evolution of mass density distribution ($g(\eta) = mf(\eta)$, $\eta = \log(m)$). The stochastic collision equation can be transferred to:

$$\begin{aligned}\frac{\partial g(\eta)}{\partial t} &= \int_{\eta_0}^{\eta_1} \frac{m^2}{(m-m')^2 m'} g(\eta-\eta') K(\eta-\eta',\eta') g(\eta') d\eta' \\ &- \int_{\eta_0}^{\infty} g(\eta) \frac{K(\eta,\eta')}{m'} g(\eta') d\eta'. \end{aligned} \tag{8.204}$$

where $\eta_1 = \log(m/2)$. Decreases of mass of i-th bin and j-th bin are given by:

$$\frac{\partial g_i^{(\mu)}}{\partial t} = -\Delta g_i^{(\mu)} K_{\mu\nu}(i,j) \frac{g_j^{(\nu)}}{m_j} \Delta\eta \tag{8.205}$$

and

$$\frac{\partial g_j^{(\mu)}}{\partial t} = -\Delta g_j^{(\nu)} K_{\mu\nu}(i,j) \frac{g_i^{(\mu)}}{m_i} \Delta\eta \tag{8.206}$$

respectively. The terms corresponds to the second term of the right-hand side of eq.(8.204). Eqs.(8.205) and (8.206) can transfer to:

$$\Delta g_i^{(\mu)} = g_i^{(\mu)} \left[1 - \exp\left(-K_{\mu\nu}(i,j) \frac{g_j^{(\nu)}}{m_j} \Delta\eta\Delta t \right) \right] \tag{8.207}$$

$$\Delta g_j^{(\nu)} = g_j^{(\nu)} \left[1 - \exp\left(-K_{\mu\nu}(i,j) \frac{g_i^{(\mu)}}{m_i} \Delta\eta\Delta t \right) \right]. \tag{8.208}$$

The sum of $\Delta g_i^{(\mu)}$ and $\Delta g_j^{(\nu)}$ corresponds to newly generated mass by collision of hydrometeors with mass of m_i and m_j. The newly generated mass ($g' = \Delta g_i^{(\mu)} + \Delta g_j^{(\nu)}$, corresponds to the first term of the right-hand side of eq.(8.204)) added k-th bin ($m_k = m_i + m_j$). Since m_k is not always bin center, newly generated mass is divided to the k-th and k+1-th bin, as follows.
The production of k-th and k+1-th bin is represented as:

$$\Delta g_k^{(\lambda)} = g_k^{\lambda} + g' - \zeta \tag{8.209}$$

$$\Delta g_{k+1}^{(\lambda)} = g_{k+1}^{\lambda} + \zeta \tag{8.210}$$

$$\begin{aligned}
\zeta &= \frac{g'}{g_k^{(\lambda)} + g'} \sum_{s=0}^{2} \frac{a_{k,s}}{(s+1)2^{k+1}} [1 - (1-2c_k)^{k+1}] \\
c_k &= \frac{m' - m_k}{m_{k+1} - m_k} \\
a_{k,0} &= -\frac{1}{24}(g_{k+1}^{(\lambda)} - 26 g_k^{(\lambda)} + g_{k-1}^{(\lambda)}) \\
a_{k,1} &= -\frac{1}{2}(g_{k+1}^{(\lambda)} - g_{k-1}^{(\lambda)}) \\
a_{k,2} &= -\frac{1}{2}(g_{k+1}^{(\lambda)} - 2 g_k^{(\lambda)} + g_{k-1}^{(\lambda)})
\end{aligned}$$

This procedure is applied for all bins of all types of hydrometeors.
In addition, for more rapid calculation, Sato et al. (2009) ' s scheme is also implemented into SCALE-RM.

Freezing

The calculation of the freezing process is based on a parameterization by Bigg (1953). The parameterization calculates number density of water ($f_c^{(w)}$) that can be frozen:

$$\begin{aligned} \frac{\partial}{\partial t} f^{(w)}(m) &= -\frac{f^{(w)}(m)}{\tau_{fr}} \\ \tau_{fr} &= \frac{\exp\left[b_{fr}(T_0 - T)\right]}{a_{fr} m} \end{aligned} \tag{8.211}$$

where $a_{fr} = 10^{-4} s^{-1}$, and $b_{fr} = 0.66^o C^{-1}$ are empirical parameters, and T_0 is 273.15 K.
Eq.(8.211) can transfer to:

$$\begin{aligned} \frac{\partial g^{(w)(m)}}{\partial t} &= -\frac{g^{(w)}(m)}{\tau_{fr}(m)} \\ \tau_{fr,i} &= \frac{\exp\left(b_{fr}(T_0 - T)\right)}{a_{fr} m} \end{aligned} \tag{8.212}$$

From this equation, the mass change of i-th bin during Δt is given as:

$$g_i^{(w)}(t + \Delta t) = g_i^{(w)} - Frz_i \tag{8.213}$$

$$\begin{cases} g_i^{(plate)}(t + \Delta t) = g_i^{(plate)} + Frz_i \quad (r_w < 200\mu m) \\ g_i^{(hail)}(t + \Delta t) = g_i^{(hail)} + Frz_i \quad (r_w > 200\mu m) \end{cases} \tag{8.214}$$

$$Frz_i = g_i^{(w)}(t)\left[1 - \exp(-\frac{\Delta t}{\tau_{fr,i}})\right]$$

As shown in eq.(8.214), the mass of liquid is transferred to plate type ice ($r_w < 200\mu m$) or hail ($r_w > 200\mu m$).

Melting

The calculation of the melting process is too simple, with all ice particles (i.e., plate, columnar, dendritic, snow, graupel and hail) melting immediately when the temperature is ¿ $T_0 = 273.15$ K. This is too simplistic to represent ice phase processes, and we will modify this method in the near future.

8.4 Surface flux

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8.4.1 Monin-Obukhov similarity

First of all, we assume that in the boundary layer 1. fluxes are constant, and 2. variables are horizontally uniform.

Relations between flux and vertical gradient are:

$$\frac{kz}{u_*}\frac{\partial u}{\partial z} = \phi_m\left(\frac{z}{L}\right), \tag{8.215}$$

$$\frac{kz}{\theta_*}\frac{\partial \theta}{\partial z} = \phi_h\left(\frac{z}{L}\right), \tag{8.216}$$

$$\frac{kz}{q_*}\frac{\partial q}{\partial z} = \phi_q\left(\frac{z}{L}\right), \tag{8.217}$$

where k is the Von Karman constant. L is the Monin-Obukhov scale height:

$$L = \frac{\theta u_*^2}{kg\theta_*}, \tag{8.218}$$

where g is gravity. The scaling velocity, u_*, temperature, θ_*, and water vapor, q_*, are defined from the vertical eddy fluxes of momentum, sensible heat, and water vapor:

$$\overline{u'w'} = -u_* u_*, \tag{8.219}$$

$$\overline{w'\theta'} = -u_* \theta_*, \tag{8.220}$$

$$\overline{w'q'} = -u_* q_*. \tag{8.221}$$

The integration between roughness length z_0 to height z of the lowest model level, eqs. (8.215) and (8.216) becomes:

$$u(z) = \frac{u_*}{k}\left\{\ln(z/z_0) - \Phi_m(z/L) + \Phi_m(z_0/L)\right\}, \tag{8.222}$$

$$\Delta\theta = R\frac{\theta_*}{k}\left\{\ln(z/z_0) - \Phi_h(z/L) + \Phi_h(z_0/L)\right\}, \tag{8.223}$$

where $\Delta\theta = \theta - \theta_0$, and

$$\Phi_m(z) = \int^z \frac{1-\phi_m(z')}{z'}dz', \tag{8.224}$$

$$\Phi_h(z) = \int^z \frac{R-\phi_h(z')}{Rz'}dz'. \tag{8.225}$$

8.4.2 Louis ’ s (1979) Model

Louis (1979) introduced a parametric model of vertical eddy fluxes.

The L becomes:

$$L = \frac{\theta u^2}{g\Delta\theta}\frac{\ln(z/z_0) - \Phi_h(z/L) + \Phi_h(z_0/L)}{\left\{\ln(z/z_0) - \Phi_m(z/L) + \Phi_m(z/L)\right\}^2}. \tag{8.226}$$

The bulk Richardson number for the layer Ri_B is:

$$Ri_B = \frac{gz\Delta\theta}{\theta u^2}, \tag{8.227}$$

and its form implies a relationship with the Monin-Obukhov scale height L. The fluxes can then be written as:

$$u_*^2 = a^2 u^2 F_m \left(\frac{z}{z_0}, Ri_B \right), \tag{8.228}$$

$$u_* \theta_* = \frac{a^2}{R} u \Delta\theta F_h \left(\frac{z}{z_0}, Ri_B \right), \tag{8.229}$$

where R is ratio of the drag coefficients for momentum and heat in the neutral limit(the turbulent Prandtl number), and

$$a^2 = \frac{k^2}{\{\ln (z/z_0)\}^2} \tag{8.230}$$

is the drag coefficient under neutral conditions.

For unstable conditions ($Ri_B < 0$), F_is ($i = m, h$) could be:

$$F_i = 1 - \frac{bRi_B}{1 + c_i \sqrt{|Ri_B|}}, \tag{8.231}$$

under the consideration that F_i must behave as $1/u$ (i.e., $\sqrt{|Ri_B|}$) in the free convection limit ($u \to 0$) and becomes 1 under neutral conditions ($Ri_B \to 0$). On the other hand, under stable conditions (Ri_b), Louis (1979) adopted the following form for F_i:

$$F_i = \frac{1}{(1 + b' Ri_B)^2}. \tag{8.232}$$

The constants are estimated as $R = 0.74$ by Businger et al. (1971), and $b = 2b' = 9.4$ by Louis (1979). By dimensional analysis:

$$c_i = C_i^* a^2 b \sqrt{\frac{z}{z_0}}, \tag{8.233}$$

and $C_m^* = 7.4, C_h^* = 5.3$, which result best fits curves.

8.4.3 Uno et al. ' s (1995) Model

Uno et al. (1995) extended the Louis Model, considering differences in roughness lengths related to momentum and temperature, i.e., z_0 and z_t, respectively.

The potential temperature difference between $z = z$ and $z = z_t$, $\Delta\theta_t$, is:

$$\begin{aligned}
\Delta\theta_t &= R\frac{\theta_*}{k} \{\ln(z_0/z_t) - \Phi_h(z_0/L) + \Phi_h(z_t/L)\} + \Delta\theta_0, \\
&= R\frac{\theta_*}{k} \ln(z_0/z_t) + \Delta\theta_0, \\
&= \Delta\theta_0 \left\{ \frac{R\ln(z_0/z_t)}{\Psi_h} + 1 \right\},
\end{aligned} \tag{8.234}$$

where $\Delta\theta_0 = \theta_z - \theta_{z_0} (= \Delta\theta)$:

$$\Psi_h = \int_{z_0}^{z} \frac{\phi_h}{z'} dz', \tag{8.235}$$

and ϕ_h is assumed to be R in the range $z_t < z < z_0$. Thus:

$$\Delta\theta_0 = \Delta\theta_t \left\{ \frac{R \ln(z_0/z_t)}{\Psi_h} + 1 \right\}^{-1}, \tag{8.236}$$

or equivalently,

$$Ri_{B0} = Ri_{Bt} \left\{ \frac{R \ln(z_0/z_t)}{\Psi_h} + 1 \right\}^{-1}. \tag{8.237}$$

From eqs. (8.228) and (8.229):

$$\Delta\theta_0 = \frac{R\theta_*}{k} \ln\left(\frac{z}{z_0}\right) \frac{\sqrt{F_m}}{F_h}, \tag{8.238}$$

while

$$\Delta\theta_0 = \frac{\theta_*}{k}\Psi_h, \tag{8.239}$$

from eqs. (8.216) and (8.235). Therefore:

$$\Psi_h = R \ln\left(\frac{z}{z_0}\right) \frac{\sqrt{F_m}}{F_h}. \tag{8.240}$$

Because Ψ_h depends on Ri_{B0}, Ri_{B0} cannot be calculated from Ri_{Bt} with eq. (8.237) directly, so numerical iteration is required to obtain Ri_{B0} [3]. Starting from Ri_{Bt} as the first estimation of Ri_{B0}, the second estimate by the Newton-Raphson iteration becomes:

$$\hat{Ri}_{B0} = Ri_{Bt} - \frac{Ri_{Bt} R \ln(z_0/z_t)}{\ln(z_0/z_t) + \hat{\Psi}_h}, \tag{8.241}$$

where $\hat{\Psi}_h$ is the estimate of Ψ_h using Ri_{Bt} instead of Ri_{B0}. Approximate values for F_m, F_h, and Ψ_h are re-calculated based on the $\hat{Ri}_{B0}$, and then $\Delta\theta_0$, and the surface fluxes u_*^2 and $u_*\theta_*$ are calculated from eqs. (8.236), (8.228), and (8.229), respectively.

R

8.4.4 Roughness length

Miller et al. (1992) provide the roughness length over the tropical ocean, based on numerical calculations by combining smooth surface values with the Charnock relation for aerodynamic roughness length and constant values for heat and moisture in accordance with Smith ' s (1988,1989) suggestions:

$$z_0 = 0.11u/\nu_* + 0.018u_*^2/g, \tag{8.242}$$

$$z_t = 0.40u/\nu_* + 1.4 \times 10^{-5}, \tag{8.243}$$

$$z_q = 0.62u/\nu_* + 1.3 \times 10^{-4}, \tag{8.244}$$

where ν_* is the kinematic viscosity of air ($\sim 1.5 \times 10^{-5}$), and z_0, z_t, and z_q are the roughness length for momentum, heat, and vapor, respectively.

[3] In the stable case, it can be solved analytically with eq. (8.232), but the solution is too complicated.

8.4.5 Discretization

All the fluxes are calculated based on the velocity at the first full-level (k=1) ($z = \Delta z/2$). The absolute velocities U are:

$$U^2_{i+\frac{1}{2},j,1} = \left\{ \frac{2(\rho u)_{i+\frac{1}{2},j,1}}{\rho_{i,j,1} + \rho_{i+1,j,1}} \right\}^2 + \left\{ \frac{(\rho v)_{i,j-\frac{1}{2},1} + (\rho v)_{i,j+\frac{1}{2},1} + (\rho v)_{i+1,j-\frac{1}{2},1} + (\rho v)_{i+1,j+\frac{1}{2},1}}{2(\rho_{i,j,1} + \rho_{i+1,j,1})} \right\}^2 + \left\{ \frac{(\rho w)_{i,j,1+\frac{1}{2}} + (\rho w)_{i+1,j,1+\frac{1}{2}}}{2(\rho_{i,j,1} + \rho_{i+1,j,1})} \right\}^2, \tag{8.245}$$

$$U^2_{i,j+\frac{1}{2},1} = \left\{ \frac{(\rho u)_{i-\frac{1}{2},j,1} + (\rho u)_{i+\frac{1}{2},j,1} + (\rho u)_{i-\frac{1}{2},j+1,1} + (\rho u)_{i+\frac{1}{2},j+1,1}}{2(\rho_{i,j,1} + \rho_{i,j+1,1})} \right\}^2 + \left\{ \frac{2(\rho v)_{i,j+\frac{1}{2},1}}{\rho_{i,j,1} + \rho_{i,j+1,1}} \right\}^2 + \left\{ \frac{(\rho w)_{i,j,1+\frac{1}{2}} + (\rho w)_{i,j+1,1+\frac{1}{2}}}{2(\rho_{i,j,1} + \rho_{i,j+1,1})} \right\}^2, \tag{8.246}$$

$$U^2_{i,j,1} = \left\{ \frac{(\rho u)_{i-\frac{1}{2},j,1} + (\rho u)_{i+\frac{1}{2},j,1}}{2\rho_{i,j,1}} \right\}^2 + \left\{ \frac{(\rho v)_{i,j-\frac{1}{2},1} + (\rho v)_{i,j+\frac{1}{2},1}}{2\rho_{i,j,1}} \right\}^2 + \left\{ \frac{(\rho w)_{i,j,1+\frac{1}{2}}}{2\rho_{i,j,1}} \right\}^2, \tag{8.247}$$

It is here of note that $(\rho w)_{i,j,\frac{1}{2}} = 0$. The potential temperatures θ are:

$$\theta_{i,j,1} = \frac{(\rho\theta)_{i,j,1}}{\rho_{i,j,1}}, \tag{8.248}$$

$$\bar{\theta}_{i+\frac{1}{2},j,1} = \frac{\theta_{i,j,1} + \theta_{i+1,j,1}}{2}, \tag{8.249}$$

$$\bar{\theta}_{i,j+\frac{1}{2},1} = \frac{\theta_{i,j,1} + \theta_{i,j+1,1}}{2}. \tag{8.250}$$

The roughness lengths, z_0, z_t, and z_q are calculated from eqs. (8.242), (8.243), and (8.244), in which the friction velocity u_* is estimated as:

$$u_* = \sqrt{C_{m0}} U, \tag{8.251}$$

where C_{m0} is a constant bulk coefficient, and we use 1.0×10^{-3} as its value.

From eq. (8.237), the Ri_{Bt}, which is the first guess of the Ri_{B0}, is:

$$Ri_{Bt} = \frac{g z_1 (\theta_1 - \theta_{sfc})}{\bar{\Theta} U^2}, \tag{8.252}$$

with the assumption that $\theta_{z_t} = \theta_{sfc}$. The estimation of $\hat{\Psi}_h$ is calculated with Ri_{Bt} from eqs. (8.240), (8.231), and (8.232). The final estimation of Ri_{B0} is obtained from eq. (8.241),and the final estimation of Ψ_h is obtained with Ri_{B0}.

Now we can calculate the bulk coefficients, C_m, C_h, and C_e for moments, heat, and vapor:

$$C_m = \frac{k^2}{\ln(z_1/z_0)} F_m(Ri_{B0}), \tag{8.253}$$

$$C_h = \frac{k^2}{R \ln(z_1/z_0)} F_h(Ri_{B0}) \left\{ \frac{R \ln(z_0/z_t)}{\Psi_h} + 1 \right\}^{-1}, \tag{8.254}$$

$$C_e = \frac{k^2}{R \ln(z_1/z_0)} F_h(Ri_{B0}) \left\{ \frac{R \ln(z_0/z_e)}{\Psi_h} + 1 \right\}^{-1}. \tag{8.255}$$

The fluxes are:

$$\overline{\rho u' w'} = -C_m U \rho u, \tag{8.256}$$

$$\overline{\rho v' w'} = -C_m U \rho v, \tag{8.257}$$

$$\overline{\rho w' w'} = -C_m U \rho w, \tag{8.258}$$

$$\overline{\rho \theta' w'} = -C_h U \{\rho\theta - \rho\theta_{sfc}\}, \tag{8.259}$$

$$\overline{\rho q' w'} = -C_e U \rho (q - q_{evap}), \tag{8.260}$$

where q_{evap} is the saturation value at the surface.

8.5 Land

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8.5.1 Land physics: slab model

The land slab model estimates soil temperature and soil moisture tendencies using a multi-layered bucket model. The soil temperature tendency equation is estimated from the 1-D vertical diffusion equation, as follows:

$$\frac{\partial T}{\partial t} = \frac{1}{\rho_L C_L} \left\{ \frac{\partial}{\partial z} \left(\kappa \frac{\partial T}{\partial z} \right) + Q \right\}, \tag{8.261}$$

where T is soil temperature (K), ρ_L is land density (kg/m^3), C_L is land heat capacity ($J/K/kg$), κ is thermal conductivity ($J/K/m/s$), and Q is external heat source ($J/m^3/s$). Eq. (8.261) is discretized as follows:

$$\frac{\Delta T_k}{\Delta t} = \frac{\nu_k}{\Delta z_k} \left(\frac{T_{k+1} - T_k}{\Delta z_{k+\frac{1}{2}}} - \frac{T_k - T_{k-1}}{\Delta z_{k-\frac{1}{2}}} \right) + \frac{Q_k}{(\rho_L C_L)_k}, \tag{8.262}$$

$$= \frac{2\nu_k}{\Delta z_k (\Delta z_{k+1} + \Delta z_k)} (T_{k+1} - T_k) - \frac{2\nu_k}{\Delta z_k (\Delta z_k + \Delta z_{k-1})} (T_k - T_{k-1}) + \frac{Q_k}{(\rho_L C_L)_k}, \tag{8.263}$$

where

$$\nu_k = \frac{\kappa}{(\rho_L C_L)_k}, \tag{8.264}$$

$$(\rho_L C_L)_k = (1 - S_{max}) C_S + S_k \rho_W C_W, \tag{8.265}$$

and S is moisture content in the k-layer (m^3/m^3), S_{max} is maximum moisture content, C_S and C_W are heat capacities of soil and water ($J/K/kg$), and ρ_W is water density (kg/m^3). The range of k is 1 to m. In this case, m is the number of the lowermost layer. Soil temperature tendency equations are implemented as follows:

$$\frac{\Delta T_1}{\Delta t} = -\frac{G_0}{(\rho_L C_L)_1 \Delta z_1} + \frac{2\nu_1}{\Delta z_1 (\Delta z_2 + \Delta z_1)} (T_2 - T_1), \tag{8.266}$$

$$\frac{\Delta T_k}{\Delta t} = -\frac{2\nu_k}{\Delta z_k (\Delta z_k + \Delta z_{k-1})} (T_k - T_{k-1}) + \frac{2\nu_k}{\Delta z_k (\Delta z_{k+1} + \Delta z_k)} (T_{k+1} - T_k), \tag{8.267}$$

$$\frac{\Delta T_m}{\Delta t} = -\frac{2\nu_m}{\Delta z_m (\Delta z_m + \Delta z_{m-1})} (T_m - T_{m-1}), \tag{8.268}$$

where G_0 is upward ground heat flux between the Earth's surface and subsurface ($J/m^2/s$). We assume that the ground heat flux under the lowermost layer is zero.

We use an implicit scheme for time integration of the land slab model. Tendency equation cofficients are summarized as follows:

$$a_1 = 0, \tag{8.269}$$

$$a_k = -\frac{2\nu_k \Delta t}{\Delta z_k(\Delta z_k + \Delta z_{k-1})}, \tag{8.270}$$

$$b_k = -\frac{2\nu_k \Delta t}{\Delta z_k(\Delta z_k + \Delta z_{k+1})}, \tag{8.271}$$

$$b_m = 0. \tag{8.272}$$

The soil temperature tendency equations are then rewritten as follows:

$$T_1^t = T_1^{t-1} - \frac{G_0 \Delta t}{\rho_L C_L \Delta z_1} + b_1(T_1^t - T_2^t), \tag{8.273}$$

$$T_k^t = T_k^{t-1} + a_k(T_k^t - T_{k-1}^t) + b_k(T_k^t - T_{k+1}^t), \tag{8.274}$$

$$T_m^t = T_m^{t-1} + a_m(T_m^t - T_{m-1}^t). \tag{8.275}$$

This simultaneous equation can be written using a matrix:

$$\begin{pmatrix} c_1 & b_1 & & & & & \\ a_2 & c_2 & b_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & a_k & c_k & b_k & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & a_{m-1} & c_{m-1} & b_{m-1} \\ & & & & & a_m & c_m \end{pmatrix} \begin{pmatrix} T_1^t \\ T_2^t \\ \vdots \\ T_k^t \\ \vdots \\ T_{m-1}^t \\ T_m^t \end{pmatrix} = \begin{pmatrix} T_1^{t-1} - \frac{G_0 \Delta t}{\rho_L C_L \Delta z_1} \\ T_2^{t-1} \\ \vdots \\ T_k^{t-1} \\ \vdots \\ T_{m-1}^{t-1} \\ T_m^{t-1} \end{pmatrix}, \tag{8.276}$$

where $c_k = 1 - a_k - b_k$. This matrix can be solved by the Thomas algorithm (tridiagonal matrix algorithm).

Soil moisture is estimated using a similar method. The soil moisture tendency equation is defined as a 1-D vertical diffusion equation, as follows:

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial z}\left(\nu \frac{\partial W}{\partial z}\right), \tag{8.277}$$

where W is soil moisture (m^3/m^3) and ν is constant water diffusivity for k (m^2/s). Eq. (8.277) is discretized as follows:

$$\frac{\Delta W_1}{\Delta t} = \frac{P - E}{\rho_W \Delta z_1} + \frac{2\nu}{\Delta z_1(\Delta z_2 + \Delta z_1)}(W_2 - W_1), \tag{8.278}$$

$$\frac{\Delta W_k}{\Delta t} = -\frac{2\nu}{\Delta z_k(\Delta z_k + \Delta z_{k-1})}(W_k - W_{k-1}) + \frac{2\nu}{\Delta z_k(\Delta z_{k+1} + \Delta z_k)}(W_{k+1} - W_k), \tag{8.279}$$

$$\frac{\Delta W_m}{\Delta t} = -\frac{2\nu}{\Delta z_m(\Delta z_m + \Delta z_{m-1})}(W_m - W_{m-1}), \tag{8.280}$$

where P is precipitation rate ($kg/m^2/s$) and E is evaporation rate ($kg/m^2/s$). As in the case of soil temperature, the tendency equations can be solved using the Thomas algorithm.

8.6 Large scale sinking

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In the DYCOMS01 experiment, large scale sinking is added to express large scale downward motion corresponding to the Hadley circulation. There is virtual convergence of motion, resulting in mass escape out of the system.

The density loss rate is constant L:

$$L = -\frac{\partial \rho w_L}{\partial z}, \tag{8.281}$$

where w_l is vertical velocity corresponding to large scale sinking. Vertical momentum with sinking is defined as follows:

$$\rho w_L = -Lz. \tag{8.282}$$

The continuous equation is now:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho(w + w_L)}{\partial z} = -L. \tag{8.283}$$

The Lagrangian conservation equation for scalar quantities is:

$$\rho\frac{\partial \phi}{\partial t} + \rho u\frac{\partial \phi}{\partial x} + \rho v\frac{\partial \phi}{\partial y} + \rho(w + w_L)\frac{\partial \phi}{\partial z} = 0, \tag{8.284}$$

When combined with eq. (8.283), this becomes:

$$\frac{\partial \rho\phi}{\partial t} + \frac{\partial \rho u\phi}{\partial x} + \frac{\partial \rho v\phi}{\partial y} + \frac{\partial \rho(w + w_L)\phi}{\partial z} = -L\phi. \tag{8.285}$$

The equation for the mixing ratio is:

$$\frac{\partial \rho Q}{\partial t} + \frac{\partial \rho Qu}{\partial x} + \frac{\partial \rho Qv}{\partial y} + \frac{\partial \rho Q(w + w_L)}{\partial z} = -LQ. \tag{8.286}$$

Note that this is identical to that for scalar quantities.

The w_L at the top boundary is not zero, while w is zero. The vertical flux $\rho w_L \phi$ at the top layer interface could be determined as that convergence of the flux canceled with $L\phi$.

Bibliography

E. K. Bigg. The formation of atmospheric ice crystals by the freezing of droplets. *Quarterly Journal of the Royal Meteorological Society*, 79:510–519, 1953.

A. Bott. A positive definite advection scheme obtained by nonlinear renormalization of the advective fluxes. *Monthly Weather Review*, 117:833–853, 1989.

A. Bott. A flux method for the numerical solution of the stochastic collection equation. *Journal of the Atmospheric Sciences*, 55:2284–2293, 1998.

A. R. Brown, S. H. Derbyshire, and P. J. Mason. Large-eddy simulation of stable atmospheric boundary layers with a revised stochastic subgrid model. *Quarterly Journal of the Royal Meteorological Society*, 120:1485–1512, 1994.

J. W. Deardorff. Stratocumulus-capped mixed layers derived from a three-dimensional model. *Boundary-Layer Meteorology*, 18:495–527, 1980.

A. Favre. Turbulence: Spacetime statistical properties and behavior in supersonic flows. *Physics of Fluids*, 26:2851–2863, 1983.

A. M. Golovin. The solution of the coagulation equation for cloud droplets in arising air current. *Bull. Acad. Sci., USSR, Geophys. Ser. X*, 5:833–853, 1963.

H. Kohler. The nucleus in and the growth of hygroscopic droplets. *Transactions of the Faraday Society*, 32:1152–1161, 1936.

A. Long. Solutions to the droplet collection equation for polynomial kernels. *Journal of the Atmospheric Sciences*, 31:1041–1052, 1974.

C.-H. Moeng and J. C. Wyngaard. Spectral analysis of large-eddy simulation of the convective boundary layer. *Journal of the Atmospheric Sciences*, 45: 3573–3587, 1988.

H. R. Pruppacher and J. D. Klett. *Microphysics of Clouds and Precipitation, 2nd edition.* Lkuwer Academic Publishers, 1997.

R. R. Rogers and M. K. Yau. *Short Course in Cloud Physics, 3rd edition.* Butterworth-Heinemann, Elsevier, Unitate States, 1989.

Y. Sato, T. Nakajima, K. Suzuki, and T. Iguchi. Application of a monte carlo integration method to collision and coagulation growth processes of hydrometeors in a bin-type model. *Journal of Geophysical Research*, 114(D9):D09215, doi:10.1029/2008JD011247, 2009.

A. Scotti, C. Meneveau, and D. K. Lilly. Generalized smagorinsky model for anisotropic grids. *Physics of Fluids A*, 5:2306–2308, 1993.

K. Suzuki. *A study on numerical modeling of cloud microphysics for calculating the particle growth process (in Japanese)*. PhD thesis, The University of Tokyo, 2006.

K. Suzuki, T. Nakajima, T. Y. Nakajima, and A. P. Khain. A study of microphysical mechanisms for correlation patterns between droplet radius and optical thickness of warm clouds with a spectral bin microphysics cloud model. *Journal of the Atmospheric Sciences*, 67(4):1126–1141, 2010.

Louis J Wicker and William C Skamarock. Time-splitting methods for elastic models using forward time schemes. *Monthly Weather Review*, 130(8):2088–2097, 2002.

Appendix A

The detal numerics

A.1 4th order central differnce

The 4th order central difference is given by

$$\frac{\partial \phi}{\partial x} = \frac{-\phi_{i+2} + 8\phi_{i+1} - 8\phi_{i-1} + \phi_{i+2}}{12\Delta x} = 0 \tag{A.1}$$

where

$$\phi_{i+2} = \phi_i + 2\Delta x \left(\frac{\partial \phi}{\partial x}\right)_i + 2\Delta x^2 \left(\frac{\partial^2 \phi}{\partial x^2}\right)_i + \frac{4\Delta x^3}{3} \left(\frac{\partial^3 \phi}{\partial x^3}\right)_i + \frac{2\Delta x^4}{3} \left(\frac{\partial^3 \phi}{\partial x^4}\right)_i + O(\Delta x^5) \tag{A.2}$$

$$\phi_{i+1} = \phi_i + \Delta x \left(\frac{\partial \phi}{\partial x}\right)_i + \frac{\Delta x^2}{2} \left(\frac{\partial^2 \phi}{\partial x^2}\right)_i + \frac{\Delta x^3}{6} \left(\frac{\partial^3 \phi}{\partial x^3}\right)_i + \frac{\Delta x^4}{24} \left(\frac{\partial^3 \phi}{\partial x^4}\right)_i + O(\Delta x^5) \tag{A.3}$$

$$\phi_i = \phi_i \tag{A.4}$$

$$\phi_{i-1} = \phi_i - \Delta x \left(\frac{\partial \phi}{\partial x}\right)_i + \frac{\Delta x^2}{2} \left(\frac{\partial^2 \phi}{\partial x^2}\right)_i - \frac{\Delta x^3}{6} \left(\frac{\partial^3 \phi}{\partial x^3}\right)_i + \frac{\Delta x^4}{24} \left(\frac{\partial^3 \phi}{\partial x^4}\right)_i + O(\Delta x^5) \tag{A.5}$$

$$\phi_{i+2} = \phi_i - 2\Delta x \left(\frac{\partial \phi}{\partial x}\right)_i + 2\Delta x^2 \left(\frac{\partial^2 \phi}{\partial x^2}\right)_i - \frac{4\Delta x^3}{3} \left(\frac{\partial^3 \phi}{\partial x^3}\right)_i + \frac{2\Delta x^4}{3} \left(\frac{\partial^3 \phi}{\partial x^4}\right)_i + O(\Delta x^5) \tag{A.6}$$

Therefore,

$$\frac{-\phi_{i+2} + 8\phi_{i+1} - 8\phi_{i-1} + \phi_{i+2}}{12\Delta x} = \left(\frac{\partial \phi}{\partial x}\right)_i + O(\Delta x^4) \tag{A.7}$$

$$\frac{(-\phi_{i+2} + 7\phi_{i+1} + 7\phi_i - \phi_{i-1}) - (-\phi_{i+1} + 7\phi_i + 7\phi_{i-1} - \phi_{i-2})}{12\Delta x} = \left(\frac{\partial \phi}{\partial x}\right)_i + O(\Delta x^4) \tag{A.8}$$

A.2 Flux Corrected Transport scheme

Equation (3.106) can be written as

$$
\begin{aligned}
(\rho q)^{n+1}_{i,j,k} &= (\rho q)^{n}_{i,j,k} - \frac{1}{\Delta x \Delta y \Delta z}\Big[\\
&+ \left[C_{i+\frac{1}{2},j,k} F^{high}_{i+\frac{1}{2},j,k} + \left(1 - C_{i+\frac{1}{2},j,k}\right) F^{low}_{i+\frac{1}{2},j,k}\right] \\
&- \left[C_{i-\frac{1}{2},j,k} F^{high}_{i-\frac{1}{2},j,k} + \left(1 - C_{i-\frac{1}{2},j,k}\right) F^{low}_{i-\frac{1}{2},j,k}\right] \\
&+ \left[C_{i,j+\frac{1}{2},k} F^{high}_{i,j+\frac{1}{2},k} + \left(1 - C_{i,j+\frac{1}{2},k}\right) F^{low}_{i,j+\frac{1}{2},k}\right] \\
&- \left[C_{i,j-\frac{1}{2},k} F^{high}_{i,j-\frac{1}{2},k} + \left(1 - C_{i,j-\frac{1}{2},k}\right) F^{low}_{i,j-\frac{1}{2},k}\right] \\
&+ \left[C_{i,j,k+\frac{1}{2}} F^{high}_{i,j,k+\frac{1}{2}} + \left(1 - C_{i,j,k+\frac{1}{2}}\right) F^{low}_{i,j,k+\frac{1}{2}}\right] \\
&- \left[C_{i,j,k-\frac{1}{2}} F^{high}_{i,j,k-\frac{1}{2}} + \left(1 - C_{i,j,k-\frac{1}{2}}\right) F^{low}_{i,j,k-\frac{1}{2}}\right] \\
&\Big]
\end{aligned}
\tag{A.9}
$$

where

$$F^{high,low}_{i+\frac{1}{2},j,k} = \Delta t \Delta y \Delta z (\rho u)_{i+\frac{1}{2},j,k} q^{high,low}_{i+\frac{1}{2},j,k} \tag{A.10}$$

$$F^{high,low}_{i,j+\frac{1}{2},k} = \Delta t \Delta z \Delta x (\rho u)_{i,j+\frac{1}{2},k} q^{high,low}_{i,j+\frac{1}{2},k} \tag{A.11}$$

$$F^{high,low}_{i,j,k+\frac{1}{2}} = \Delta t \Delta x \Delta y (\rho u)_{i,j,k+\frac{1}{2}} q^{high,low}_{i,j,k+\frac{1}{2}} \tag{A.12}$$

The anti-diffusive flux are defined as

$$A_{i+\frac{1}{2},j,k} = F^{high}_{i+\frac{1}{2},j,k} - F^{low}_{i+\frac{1}{2},j,k} \tag{A.13}$$

$$A_{i,j+\frac{1}{2},k} = F^{high}_{i,j+\frac{1}{2},k} - F^{low}_{i,j+\frac{1}{2},k} \tag{A.14}$$

$$A_{i,j,k+\frac{1}{2}} = F^{high}_{i,j,k+\frac{1}{2}} - F^{low}_{i,j,k+\frac{1}{2}} \tag{A.15}$$

Equation (A.9) can be rewritten as

$$
\begin{aligned}
(\rho q)^{n+1}_{i,j,k} &= (\rho q)^{n}_{i,j,k} - \frac{1}{\Delta x \Delta y \Delta z}\Big[\\
&+ \left[F^{low}_{i+\frac{1}{2},j,k} + C_{i+\frac{1}{2},j,k} A_{i+\frac{1}{2},j,k}\right] \\
&- \left[F^{low}_{i-\frac{1}{2},j,k} + C_{i-\frac{1}{2},j,k} A_{i-\frac{1}{2},j,k}\right] \\
&+ \left[F^{low}_{i,j+\frac{1}{2},k} + C_{i,j+\frac{1}{2},k} A_{i,j+\frac{1}{2},k}\right] \\
&- \left[F^{low}_{i,j-\frac{1}{2},k} + C_{i,j-\frac{1}{2},k} A_{i,j-\frac{1}{2},k}\right] \\
&+ \left[F^{low}_{i,j,k+\frac{1}{2}} + C_{i,j,k+\frac{1}{2}} A_{i,j,k+\frac{1}{2}}\right] \\
&- \left[F^{low}_{i,j,k-\frac{1}{2}} + C_{i,j,k-\frac{1}{2}} A_{i,j,k-\frac{1}{2}}\right] \\
&\Big]
\end{aligned}
\tag{A.16}
$$

In practice, we calculate Eq.(A.16) by the following steps:

1. The tentative values are calculated by using the low order flux:

$$
\begin{aligned}
(\rho q)^{\dagger}_{i,j,k} &= (\rho q)^{n}_{i,j,k} \\
&- \frac{1}{\Delta x \Delta y \Delta z}\left[+F^{low}_{i+\frac{1}{2},j,k} - F^{low}_{i-\frac{1}{2},j,k} + F^{low}_{i,j+\frac{1}{2},k} - F^{low}_{i,j-\frac{1}{2},k} + F^{low}_{i,j,k+\frac{1}{2}} - F^{low}_{i,j,k-\frac{1}{2}}\right]
\end{aligned}
\tag{A.17}
$$

2. Allowable maximum and minimum values are calculated:

$$\begin{aligned}
(\rho q)^{\max}_{i,j,k} &= \max[\\
&\max((\rho q)^{\dagger}_{i,j,k}, (\rho q)^{n}_{i,j,k}), \\
&\max((\rho q)^{\dagger}_{i-1,j,k}, (\rho q)^{n}_{i-1,j,k}), \\
&\max((\rho q)^{\dagger}_{i+1,j,k}, (\rho q)^{n}_{i+1,j,k}), \\
&\max((\rho q)^{\dagger}_{i,j-1,k}, (\rho q)^{n}_{i,j-1,k}), \\
&\max((\rho q)^{\dagger}_{i,j+1,k}, (\rho q)^{n}_{i,j+1,k}), \\
&\max((\rho q)^{\dagger}_{i,j,k-1}, (\rho q)^{n}_{i,j,k-1}), \\
&\max((\rho q)^{\dagger}_{i,j,k+1}, (\rho q)^{n}_{i,j,k+1}) \\
&] \qquad (A.18)
\end{aligned}$$

$$\begin{aligned}
(\rho q)^{\min}_{i,j,k} &= \min[\\
&\min((\rho q)^{\dagger}_{i,j,k}, (\rho q)^{n}_{i,j,k}), \\
&\min((\rho q)^{\dagger}_{i-1,j,k}, (\rho q)^{n}_{i-1,j,k}), \\
&\min((\rho q)^{\dagger}_{i+1,j,k}, (\rho q)^{n}_{i+1,j,k}), \\
&\min((\rho q)^{\dagger}_{i,j-1,k}, (\rho q)^{n}_{i,j-1,k}), \\
&\min((\rho q)^{\dagger}_{i,j+1,k}, (\rho q)^{n}_{i,j+1,k}), \\
&\min((\rho q)^{\dagger}_{i,j,k-1}, (\rho q)^{n}_{i,j,k-1}), \\
&\min((\rho q)^{\dagger}_{i,j,k+1}, (\rho q)^{n}_{i,j,k+1}) \\
&] \qquad (A.19)
\end{aligned}$$

3. Several values for the flux limiter are calculated:

$$\begin{aligned}
P^{+}_{i,j,k} &= -\min(0, A_{i+\frac{1}{2},j,k}) + \max(0, A_{i-\frac{1}{2},j,k}) \\
&\quad -\min(0, A_{i,j+\frac{1}{2},k}) + \max(0, A_{i,j-\frac{1}{2},k}) \\
&\quad -\min(0, A_{i,j,k+\frac{1}{2}}) + \max(0, A_{i,j,k-\frac{1}{2}}) \qquad (A.20) \\
P^{-}_{i,j,k} &= -\max(0, A_{i+\frac{1}{2},j,k}) + \min(0, A_{i-\frac{1}{2},j,k}) \\
&\quad -\max(0, A_{i,j+\frac{1}{2},k}) + \min(0, A_{i,j-\frac{1}{2},k}) \\
&\quad -\max(0, A_{i,j,k+\frac{1}{2}}) + \min(0, A_{i,j,k-\frac{1}{2}}) \qquad (A.21) \\
& \qquad (A.22)
\end{aligned}$$

$$Q^{+}_{i,j,k} = \left[(\rho q)^{\max}_{i,j,k} - (\rho q)^{\dagger}_{i,j,k}\right] \Delta x \Delta y \Delta z \qquad (A.23)$$

$$Q^{-}_{i,j,k} = \left[(\rho q)^{\dagger}_{i,j,k} - (\rho q)^{\min}_{i,j,k}\right] \Delta x \Delta y \Delta z \qquad (A.24)$$

$$R^{+}_{i,j,k} = \begin{cases} \min(1, Q^{+}_{i,j,k}/P^{+}_{i,j,k}) & \text{if } P^{+}_{i,j,k} > 0 \\ 0 & \text{if } P^{+}_{i,j,k} = 0 \end{cases} \qquad (A.25)$$

$$R^{-}_{i,j,k} = \begin{cases} \min(1, Q^{-}_{i,j,k}/P^{-}_{i,j,k}) & \text{if } P^{-}_{i,j,k} > 0 \\ 0 & \text{if } P^{-}_{i,j,k} = 0 \end{cases} \qquad (A.26)$$

4. The flux limters at the cell wall are calculated:

$$C_{i+\frac{1}{2},j,k} = \begin{cases} \min(R^+_{i+1,j,k}, R^-_{i,j,k}) & \text{if } A^-_{i+\frac{1}{2},j,k} \geq 0 \\ \min(R^+_{i,j,k}, R^-_{i+1,j,k}) & \text{if } A^-_{i+\frac{1}{2},j,k} < 0 \end{cases} \tag{A.27}$$

$$C_{i,j+\frac{1}{2},k} = \begin{cases} \min(R^+_{i,j+1,k}, R^-_{i,j,k}) & \text{if } A^-_{i,j+\frac{1}{2},k} \geq 0 \\ \min(R^+_{i,j,k}, R^-_{i,j+1,k}) & \text{if } A^-_{i,j+\frac{1}{2},k} < 0 \end{cases} \tag{A.28}$$

$$C_{i,j,k+\frac{1}{2}} = \begin{cases} \min(R^+_{i,j,k+1}, R^-_{i,j,k}) & \text{if } A^-_{i,j,k+\frac{1}{2}} \geq 0 \\ \min(R^+_{i,j,k}, R^-_{i,j,k+1}) & \text{if } A^-_{i,j,k+\frac{1}{2}} < 0 \end{cases} \tag{A.29}$$

Appendix B

Notation

Table B.1: Notation of symbols

ρ	total density	kg/m^3
q_d	mass concentration of dry air	$-$
q_v	mass concentration of water vapor	$-$
q_l	mass concentration of liquid water	$-$
q_s	mass concentration of solid water	$-$
t	time	s
$\mathbf{u}$	velocity of air flow	m/s
w_l	relative velocity of liquid water to the gas	m/s
w_s	relative velocity of solid water to the gas	m/s
$\mathrm{DIFF}\,[x]$	Diffusion term by turbulene	$kg/m^3\,[x]\,/s$
S_v	source term of water vapor	$kg/m^3/s$
S_l	source term of liquid water	$kg/m^3/s$
S_s	source term of solid water	$kg/m^3/s$
p	pressure	N/m^2
g	gravitational acceraration	9.8 m/s^2
f_l	drag force due to water loading by liquid water	$kg/m^2/s^2$
f_s	drag force due to water loading by solid water	$kg/m^2/s^2$
$\mathbf{e_z}$	vertical unit vector (upward)	$-$
R_d	gas constant for dry air for uint mass	J/kg
R_v	gas constant for water vapor for uint mass	J/kg
T	temperature	K
Q_d	diabatic heating due to physical processes for dry air	$J/m^3/s$
Q_v	diabatic heating due to physical processes for water vapor	$J/m^3/s$
Q_l	diabatic heating due to physical processes for liquid water	$J/m^3/s$
Q_s	diabatic heating due to physical processes for solid water	$J/m^3/s$
e_d	internal energy for dry air	J/kg
e_v	internal energy for water vapor	J/kg
e_l	internal energy for liquid water	J/kg
e_s	internal energy for solid water	J/kg
e	total internal energy	J/kg
c_{vd}	specific heat at constant volume for dry air	$J/kg/K$
c_{vv}	specific heat at constant volume for water vapor	$J/kg/K$
c_{pd}	specific heat at constant pressure for dry air	$J/kg/K$
c_{pv}	specific heat at constant pressure for water vapor	$J/kg/K$
c_l	specific heat for liquid water	$J/kg/K$
c_s	specific heat for solid water	$J/kg/K$
p_{00}	standard pressure	1000.0 Pa
θ_d	potential temperature for dry air	K
θ	total potential temperature	K

Appendix C

Variables in the source code

Table C.1: Variables in atmos/mod_atmos_dyn_fent_fct.f90.

DENS(k,i,j)	$\rho_{i,j,k}$
MOMZ(k,i,j)	$(\rho w)_{i,j,k+\frac{1}{2}}$
MOMX(k,i,j)	$(\rho u)_{i+\frac{1}{2},j,k}$
MOMY(k,i,j)	$(\rho v)_{i,j+\frac{1}{2},k}$
RHOT(k,i,j)	$(\rho\theta)_{i,j,k}$
QTRC(k,i,j,iq)	$q_{i,j,k}$
PRES(k,i,j)	$p_{i,j,k}$
VELZ(k,i,j)	$\bar{w}_{i,j,k+\frac{1}{2}}$
VELX(k,i,j)	$\bar{u}_{i+\frac{1}{2},j,k}$
VELY(k,i,j)	$\bar{v}_{i,j+\frac{1}{2},k}$
POTT(k,i,j)	$\theta_{i,j,k}$
QDRY(k,i,j)	q_d
Rtot(k,i,j)	R^*
num_diff(k,i,j)	$F_{i+\frac{1}{2}}$
qflx_hi(k,i,j)	$\bar{q}^{high}$
qflx_lo(k,i,j)	$\bar{q}^{low}$
qjpls(k,i,j)	$Q^+_{i,j,k}$
qjmns(k,i,j)	$Q^-_{i,j,k}$
pjpls(k,i,j)	$P^+_{i,j,k}$
pjmns(k,i,j)	$P^-_{i,j,k}$
rjpls(k,i,j)	$R^+_{i,j,k}$
rjmns(k,i,j)	$R^-_{i,j,k}$

Table C.2: Variables in atmos/mod_atmos_phy_tb_smg.f90.

tke(k,i,j)	TKE
nu(k,i,j), nu_*(k,i,j)	ν_{SGS}
Ri(k,i,j)	Ri
Pr(k,i,j)	Pr
S33_*(k,i,j)	S_{33}
S11_*(k,i,j)	S_{11}
S22_*(k,i,j)	S_{22}
S31_*(k,i,j)	S_{31}
S12_*(k,i,j)	S_{12}
S23_*(k,i,j)	S_{23}
qflx_sgs(k,i,j)	$\bar{\rho}\tau_{ij}$, $\bar{\rho}\tau^*_{ij}$